REGION OF CONVERGENCE OF DIRICHLET SERIES WITH COMPLEX EXPONENTS[1]

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Consider the Dirichlet series

$$\sum_{n=1}^{\infty} c_n e^{-\lambda_n s} \tag{1}$$

with complex exponents λ_n. Hille [1] has given a geometrical description, which was somewhat simplified later by Valiron [2], of a convex maximal region outside of which the series does not converge and in the interior of which, in case

$$\lim_{n\to\infty} (\log n)/\lambda_n = 0, \tag{2}$$

the series converges absolutely. In this paper we give a simpler definition (5) of the maximal region, a generalization (4) of the Cauchy-Hadamard formula to give the distance from an interior point to the frontier of this region, and, as an application of this formula, a generalization (Theorem 2) of Hille's result. This contains a well known relation between the abscissas of simple and absolute convergence for Dirichlet series with positive exponents.

It will be convenient to introduce the notation

$$d(n, s) = - |\lambda_n|^{-1} \log |c_n e^{-\lambda_n s}|. \tag{3}$$

The geometrical significance of the number $d(n, s)$ is that it is the algebraic distance from the point s of the complex s-plane to the line L_n on which the absolute value of the term $c_n e^{-\lambda_n s}$ is 1. Let

$$d(s) = \liminf_{n\to\infty} d(n, s) \tag{4}$$

and

$$D = \{s : d(s) \geqq 0\}. \tag{5}$$

It is easy to see that the region D is convex and closed and that the series does not converge outside of D. For if s_1 and s_2 belong to D and if $s = ts_1 + (1-t)s_2$, where $0 \leqq t \leqq 1$, then

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$$d(n, s) = td(n, s_1) + (1 - t)d(n, s_2)$$

and hence

$$d(s) \geqq td(s_1) + (1 - t)d(s_2) \geqq 0$$

and thus D is convex. To see that D is closed, suppose that s is a limit point of points of D. Then, given any $\epsilon > 0$, there is a point s_ϵ of D such that $|s - s_\epsilon| < \epsilon$ or $d(n, s) > d(n, s_\epsilon) - \epsilon$ or

$$d(s) \geqq d(s_\epsilon) - \epsilon \geqq -\epsilon,$$

which shows that s belongs to D. Suppose, finally, that s does not belong to D or, in other words that $d(s) < 0$. Then there exists an $\epsilon > 0$ and a sequence of positive integers n_i such that $d(n_i, s) < -\epsilon$ for $i = 1, 2 \cdots$. Since

$$\left| c_{n_i} e^{-\lambda_{n_i} s} \right| = e^{-|\lambda_{n_i}| d(n_i, s)} > e^{|\lambda_{n_i}| \epsilon} > 1,$$

we see that the series does not converge at the point s.

Slightly less obvious is the proof of the following:

THEOREM 1. *If s belongs to D, then the distance from s to the frontier of D is $d(s)$.*

Let us denote the distance in question by K and $d(s)$ by R. We shall show first that $K \geqq R$. Since this is trivially true for $R = 0$, assume $R > 0$. Given any $\epsilon > 0$, we have $d(n, s) > R - \epsilon$ and hence $d(n, s') > \epsilon$ for n sufficiently large and $|s - s'| < R - 2\epsilon$, which shows that s' belongs to D if $|s - s'| < R$, and hence that $K \geqq R$.

On the other hand, if we denote the foot of the perpendicular from s on the line L_n by s_n, since there exists a sequence of positive integers n_i such that $\lim_{i\to\infty} d(n_i, s) = R$, we have

$$\lim_{i\to\infty} \left| s - s_{n_i} \right| = R.$$

Now the sequence of points s_{n_i} has at least one limit point s_0 and it is impossible that $|s - s_0| > R$. Clearly, also, $d(s_0) \leqq 0$.

Suppose s_0 is an interior point of D. Then there exists an equilateral triangle with center at s_0 and vertices v_1, v_2, and v_3 which belong to D. Denote the length of the medians by m. If n is sufficiently large, then $d(n, v_i) > -m/4$ for $i = 1, 2, 3$, and hence $d(n, s_0) > m/12$, or $d(s_0) > 0$. Since this is a contradiction, s_0 can not be an interior point. But, from the first paragraph of this proof and the fact that D is closed, it follows that s_0 belongs to D. Thus s_0 is a frontier point of D and hence $K \leqq R$, which completes the proof.

An application of this theorem is the following.

THEOREM 2. *The series converges absolutely to an analytic function at every point of D whose distance from the frontier of D is greater than*

$$L = \limsup_{n\to\infty} (\log n)/|\lambda_n|. \tag{6}$$

Suppose the distance from a point s of D to the frontier of D is greater than L. Then there exists a number $\epsilon > 0$ sufficiently small that $d(n, s) > L+3\epsilon$ for n sufficiently large and hence $d(n, s') > L+2\epsilon$ if $|s-s'| \leqq \epsilon$. Since $|\lambda_n| > (L+\epsilon)^{-1} \log n$ for n sufficiently large,

$$|c_n e^{-\lambda_n s'}| = e^{-|\lambda_n| d(n,s')} < n^{-(L+2\epsilon)/(L+\epsilon)}$$

from which we conclude that the series converges absolutely and uniformly on the disc $|s-s'| \leqq \epsilon$.

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