Without significant loss of generality we may take $\mu_1 = 1$. Determining $g(u)$, $A(y)$, $\psi(t)$ from the differential equations (14)(i), (ii) and the recurrence relation (12)(i) leads to known generating functions.

For $\alpha - \beta = 0$ we have the ultraspherical polynomials with known generating functions of Appell type.

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APPROXIMATION BY FAMILIES OF FUNCTIONS[1]

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Let F be a set of real functions on the interval $a \leqq x \leqq b$ and let $g(x)$ be a real and continuous function on the same interval. The problem of finding a function in F which approximates g best according to some criterion has been extensively studied, especially in the two cases when F is the set of all real polynomials of degree less than n or a fixed set of trigonometric functions [1; 2; 4, pp. 40–41]. We investigate the problem for n-parameter families of functions [5] with reference to some of the usual approximation criteria.

An n-parameter family F of functions on the interval $a \leqq x \leqq b$ is a set of real continuous functions f such that for every set of points (x_i, y_i) $(i = 1, \cdots, n)$ with $a \leqq x_1 < \cdots < x_n \leqq b$ there exists exactly one f in F with $f(x_i) = y_i$ $(i = 1, \cdots, n)$.

An n-parameter family is linear if it is a real vector space, i.e., if there exist n functions $f_1, \cdots, f_n$ such that every f may be expressed as a linear combination $f = a_1 f_1 + \cdots + a_n f_n$, where $a_1, \cdots, a_n$ are real.

For $k \geqq 1$ the modulus of approximation $M^{(k)}$ of g by f is defined as

$$M^{(k)}(f) = \int_a^b |f - g|^k dx \qquad (k < \infty),$$

$$M^{(\infty)}(f) = \max |f - g| .$$

Further a function f_0 in F will be called a best k-approximant to g if

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$$M^{(k)}(f_0) = \underset{f \in F}{\text{g.l.b.}}\, M^{(k)}(f).$$

The existence and uniqueness of a best ∞-approximant was demonstrated in [5, Theorems 7 and 9].

THEOREM 1. *A best k-approximant always exists.*

Let $f_1, f_2, \cdots$ be a sequence of functions (in the n-parameter family F) such that $\lim_{i\to\infty} M^{(k)}(f_i) = m$, where $m = \text{g.l.b. } M^{(k)}(f)$. Let $d = (b-a)/3(n+1)$, and let $x_j = a + 3jd$ $(j = 1, 2, \cdots, n)$. On the interval $x_j - d \leqq x \leqq x_j + d$, let the minimum of $|f_i - g|$ occur at x_{ij} and set $y_{ij} = f_i(x_{ij})$. Since the sequence $\{M^{(k)}(f_i)\}$ $(i = 1, 2, \cdots)$ is bounded the set of values y_{ij} is bounded. Hence for each j, the points (x_{ij}, y_{ij}) have a limit point (x_j, y_j). Hence there is a subsequence $\{f_{i'}\}$ of the sequence $\{f_i\}$ such that $\lim_{i'\to\infty} (x_{i'j}, y_{i'j}) = (x_j, y_j)$ $(j = 1, \cdots, n)$. Consequently, if f is the function in F through $(x_1, y_1), \cdots, (x_n, y_n)$, $\lim_{i'\to\infty} f_{i'} = f$ uniformly [5, Theorem 5, p. 460]; thus $M^{(k)}(f_{i'}) \to M^{(k)}(f)$, and so f is a best k-approximant.

THEOREM 2. *A best k-approximant is unique if F is linear and* $k > 1$.

When k is an even integer, the proof of Pólya [3] for the special case of polynomials is also effective here.

In general we use the strict convexity of $y = |x|^k$. If $f_1 \neq f_2$ and $f = (f_1 + f_2)/2$ then $|f - g|^k \leqq (|f_1 - g|^k + |f_2 - g|^k)/2$, with inequality at some point; thus, since the functions are continuous $M^{(k)}(f) < [M^{(k)}(f_1) + M^{(k)}(f_2)]/2$. Hence the best approximant is unique. Were $k = 1$ all that we could say on the basis of this proof is that $(f_1 - g)(f_2 - g) \geqq 0$ if f_1 and f_2 are best 1-approximants.

THEOREM 3. *If* $n = 1$, *a best k-approximant need not be unique for* $k \neq \infty$. *Also it need not cross g.*

We give an example for $k = 1$. Let $a = 0$, $b = 1$. Let $g(x) = 0$. We designate by f_r that f in F for which $f(0) = r$, i.e., by its y-intercept. Let f_r for $r \leqq -1$ be the line segment of slope 1; for $-1 \leqq r \leqq -1/2$ the pair of line segments, one joining $(0, r)$ to $(s, s+r)$ having slope 1 and the other $(s, s+r)$ to $(1, 3-2s+r)$ having slope 3, where $s = 3 + 2r - [3(r^2 + 3r + 2)]^{1/2}$; and for $-1/2 \leqq r$, $f_r = f_{-1/2} + r + 1/2$. Then all f_r $(-1 \leqq r \leqq -1/2)$ are best 1-approximants to g and f_{-1} does not cross g. Only a slight change in the definition of s is needed to take care of other values of k.

THEOREM 4. *Let* f_k *be a sequence of best k-approximants where* $k \to \infty$. *Then* $\lim_{k\to\infty} f_k = f_\infty$.

The proof of Pólya [3] for the case of polynomials actually takes care of the general case after we have proved that there is a subsequence of the f_k which approaches a limit f^* uniformly. This can be done as in the proof of Theorem 1 after we have shown that $M^{(k)}(f_k)$ is bounded. But this fact was also proved by Pólya as follows:

$$M^{(k)}(f_k) \leqq M^{(k)}(f_\infty) \leqq [M^{(\infty)}(f_\infty)]^k(b-a).$$

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