ON HOLONOMY COVERING SPACES

LOUIS AUSLANDER[1]

Introduction. In [1] L. Markus and the author introduced the concept of holonomy covering spaces for flat affinely connected manifolds or what are also called locally affine spaces. We also proved in [1] that the holonomy covering space of a complete n dimensional locally affine space must be some n dimensional cylinder, i.e., $T^i \times F^{n-i}$, $i=0, \cdots, n$, where T^i denotes the locally affine i dimensional torus and E^{n-i} denotes the $n-i$ dimensional affine space. It is the purpose of this paper to prove that all these holonomy covering spaces are actually realized. To be more explicit, we shall construct n locally affine connections A_i^n, $i=1, \cdots, n$, on the n dimensional torus such that the holonomy covering space of A_i^n is $T^i \times F^{n-i}$, $i=1, \cdots, n$.

1. **General considerations.** Let E^n be the n dimensional affine space and let Π be a subgroup of the group of affine transformations of E^n. Let us assume that Π acts on E^n without fixed points and properly discontinuously, i.e., without accumulation points. We shall define two points x and y in E^n to be equivalent if there exists a $g \in \Pi$ such that $g(x)=y$. Then the equivalence classes of E^n with the induced topology form a manifold which we shall denote by E^n/Π. The class of such manifolds is precisely the class of complete locally affine spaces. If we let E^n have $(x_1, \cdots, x_n)$ as coordinate systems, then the elements of Π may be represented as $(n+1)\times(n+1)$ matrices whose last row consists of all zeros except in the $n+1$ entry where it is one. Let $g \in \Pi$. We may associate to g the $n \times n$ matrix consisting of the first n rows and columns of the matrix representation of g. Call this mapping H. Then it is an easy consequence of the discussion in [1] that $H(\Pi)$ is precisely a matrix representation for the holonomy group of the space E^n/Π with its induced affine connection. If $k(\Pi)$ denotes the kernel of this homomorphism H, then $k(\Pi)$ is the fundamental group of the holonomy covering space of E^n/Π. Hence $k(\Pi)$ is a free abelian group on at most n generators. Under the assumption that $k(\Pi)$ has q generators, the holonomy covering space of E^n/Π is $T^q \times E^{n-q}$, $q=1, \cdots, n$.

Hence to determine the desired affine connections A_i^n, $i=1, \cdots, n$,

---

Received by the editors July 29, 1955.

[1] During part of the time the author was writing this paper he was a National Science Foundation Post-Doctoral Fellow.

we have to construct groups Π_i^n, $i=1, \cdots, n$, which have the following two properties:

1. $E^n/\Pi_i^n \approx T^n$.
2. $k(\Pi_i^n)$ has $n-i$ generators, $i=1, \cdots, n$.

2. **Reduction of the problem.** We shall now show that the knowledge of Π_1^2, Π_2^2, Π_3^3 is sufficient for constructing Π_i^n, $i=1, \cdots, n$.

CASE 1. $i=1$ and $n \geqq 2$.

Let $\Pi_1^2(1)$ acting on $(x_1, \cdots, x_n)$ act as Π_1^2 on x_1 and x_2 and the identity mapping on the other coordinates. Let P_j be the unit translation in the j direction. Let Π_1^n be the group generated by $\Pi_1^2(1)$ and P_j, $3 \leqq j \leqq n$.

CASE 2. i even and $n \geqq 2$.

Let $2\alpha = i$. We consider the affine transformation $\Pi_2^2(q)$, $q=1, \cdots, \alpha$, which is the identity transformation except on x_{2q-1}, x_{2q} which it transforms as Π_2^2. Let Π_i^n be the group generated by $\Pi_2^2(q)$, $q=1, \cdots, \alpha$, and P_j, $i+1 \leqq j \leqq n$.

CASE 3. i odd and $n \geqq 2$.

Let $i-3=2\alpha$. Let $\Pi_3^3(i)$ be the identity transformation except on $x_{2\alpha+1}$, $x_{2\alpha+2}$, $x_{2\alpha+3}$ which it transforms as Π_3^3. Let Π_i^n be generated by $\Pi_3^3(i)$, $\Pi_2^2(q)$, $q=1, \cdots, \alpha$, and P_j, $i+1 \leqq j \leqq n$.

It is now a trivial matter to prove that Π_i^n has the desired properties.

From the work of Kuiper we see that we must take Π_1^2 as the group generated by

$$\begin{aligned} x' &= x + y \\ y' &= y + 1 \end{aligned} \quad \text{and} \quad \begin{aligned} x' &= x + 1, \\ y' &= y \end{aligned}$$

and Π_2^2 as the group generated by

$$\begin{aligned} x' &= x + y \\ y' &= y + 1 \end{aligned} \quad \text{and} \quad \begin{aligned} x' &= x + qy + \lambda, \\ y' &= y + q \end{aligned}$$

where $\lambda \neq q(q-1)/2$ and q is irrational.

3. **On Π_3^3.** Let g_1, g_2, g_3 be the transformations

$$\begin{aligned} x' &= x + y, \\ y' &= y + 1, \\ z' &= z, \end{aligned} \qquad \begin{aligned} x' &= x + z, \\ y' &= y, \\ z' &= z + 1, \end{aligned} \qquad \begin{aligned} x' &= x + qy + qz + k, \\ y' &= y + q, \\ z' &= z + q, \end{aligned}$$

respectively, where $0 < q < 1/8$ is irrational and $k \geqq 4$. Let Π_3^3 be the group generated by g_1, g_2, and g_3. It is a perfectly straightforward matter to prove that Π_3^3 is abelian and that the general term $g_1^n g_2^m g_3^s$ of Π_3^3 is given by

$$
\begin{aligned}
x' &= x + (n + sq)y + (m + sq)z + \lambda,\\
y' &= y + n + sq, \qquad (1)\\
z' &= z + m + sq,
\end{aligned}
$$

where

$$\lambda = sk + nsq + msq + 2q^2 f(s) + f(n) + f(m)$$

and

$$f(n) = \begin{cases} \sum_{i=1}^{n-1} i, & n > 0, \\ \sum_{i=1}^{-n} i, & n < 0. \end{cases}$$

This can be simplified by noting that $f(n) = (n^2 - n)/2$ for all n. Hence

$$\lambda = sk + q^2(s^2 - s) + 1/2(m^2 - m) + 1/2(n^2 - n) + nsq + msq$$

or

$$\lambda = 1/2(n + sq)^2 + 1/2(m + sq)^2 + s(k - q^2) - 1/2n - 1/2m. \qquad (2)$$

LEMMA 1. *Π_3^3 acts on E^3 without fixed points.*

PROOF. Let us assume there is a fixed point. Then by equation (1) we must have $n + sq = 0$. But this is impossible since q is irrational.

LEMMA 2. *Π_3^3 acts properly discontinuously on E^3.*

PROOF. Let us assume that this is false. Then there exist (n_i, m_i, s_i), $i = 1, 2, \cdots$, and a point (x_0, y_0, z_0) such that $g_1^{n_i} g_2^{m_i} g_3^{s_i}(x_0, y_0, z_0)$ is a Cauchy sequence in E^n. Hence by equation (1), $n_i + s_i q$ and $m_i + s_i q$, $i = 1, 2, \cdots$, must be Cauchy sequences. But since q is irrational this implies that $\lim_{i\to\infty} |n_i| = \infty$, $\lim_{i\to\infty} |m_i| = \infty$, and $\lim_{i\to\infty} |s_i| = \infty$. Further s_i and n_i must have opposite signs as must s_i and m_i. But by equation 2 we see that if x_i is the x coordinate of $g_1^{n_i} g_2^{m_i} g_3^{s_i}(x_0, y_0, z_0)$, then $\lim_{i\to\infty} |x_i| = \infty$. This contradiction completes the proof of this lemma.

We shall use a bracket about a set of points in E^3 to denote the point set determined by the closed simplex which has these points as vertices. Let 0 denote the origin in E^3. Then it is straightforward to verify that

$$[0,\ g_i(0),\ g_ig_j(0),\ g_ig_jg_k(0)]$$

are nondegenerate simplices in E^3, where i, j, k form a permutation of 1, 2, 3. Let F denote the union of the above sets under all the permutations of 1, 2, 3. It will now be our object to prove that F is a fundamental domain for Π_3^3.

Let $F(2)=[0,\ g_1(0),\ g_1g_3(0)]\ [0,\ g_3(0);\ g_1g_3(0)]$ and let $R_0(2) = \bigcup g_1^n g_3^s F(2)$ for all integers n and s. By cyclic permutation we may define $F(i)$ and $R_0(i)$, $i=1, 2, 3$.

LEMMA 3. *The projection p of $R_0(2)$ perpendicular to the plane $z=0$ is a one-to-one mapping of $R_0(2)$ onto the plane $z=0$.*

PROOF. The perpendicular projection of a two-simplex onto a plane is one-to-one unless the image of the simplex is a line. Now the vertices of $R_0(2)$ map into the points with coordinates $(\lambda(n,\ s,\ m=0),\ n+sq,\ 0)$. Consider the group $\Pi^1(2)$ of transformations of the plane $z=0$ generated by $\bar{g}_1$ and $\bar{g}_3$, where $\bar{g}_1$ and $\bar{g}_3$ are given by

$$\begin{aligned} x' &= x + y, & x' &= x + qy + k, \\ y' &= y + 1, & y' &= y + q, \end{aligned}$$

respectively. Now $p(g_1^n g_3^s(0)) = \bar{g}_1^n \bar{g}_3^s(0)$. Hence the range of $g_1^n g_3^s[0,\ g_i(0),\ g_ig_j(0)]$ under p is $\bar{g}_1 \bar{g}_3^s[0,\ \bar{g}_i(0),\ \bar{g}_i\bar{g}_j(0)]$, where i, j form a permutation of 1 and 2. Since the plane $z=0$ under the action of $\Pi^1(2)$ is a torus we must have that $\bar{g}_1^n \bar{g}_3^s[0,\ \bar{g}_i(0),\ \bar{g}_i\bar{g}_j(0)]$ are never degenerate. Hence p is a one-to-one mapping of each of the simplices of $R_0(2)$. Further p must be one-to-one in the large since $[0,\ \bar{g}_1(0),\ \bar{g}_1\bar{g}_3(0)] \cup [0,\ \bar{g}_3(0),\ \bar{g}_1\bar{g}_3(0)]$ is a fundamental domain for the plane $z=0$ under the action of $\Pi^1(2)$. This also implies that the mapping p is onto.

We have also shown

LEMMA 4. *If $\Pi(2)$ is the group generated by g_1 and g_3 then $R_0(2)/\Pi(2)$ is homeomorphic to a two-dimensional torus and $F(2)$ is a fundamental domain for $\Pi(2)$.*

Let $R_s(i) = g_i^s R_0(i)$.

LEMMA 5. $R_0(2) \cap R_1(2) = 0$.

PROOF. Let us assume this lemma is false. Then there exist points f_1 and $f_2 \in F(2)$ such that $g_1^n g_3^s f_1 = g_2 g_1^{n'} g_3^{s'} f_2$. Hence $g_2 f_2 = g^{n-n'} g_3^{s-s'} f_1$. But it is trivial to see that the $p[g_2(F(2))]$ meets $R_0(2)$. This contradicts Lemma 3 unless $n=n'$ and $s=s'$. But then $g_2 f_1 = f_2$ or $g_2 F(2) \cap F(2)$ is not empty. But this is false.

LEMMA 6. *$R_m(2)$ separates E^3 and $R_m(2)$ lies between $R_{m-1}(2)$ and $R_{m+1}(2)$ for all m.*

This follows from Lemmas 3 and 5 and the fact that Π_3^3 is orientation preserving.

LEMMA 7. *Let f_1 and f_2 be between $R_0(2)$ and $R_1(2)$. If $g \in \Pi_3^3$ exists such that $g(f_1) = f_2$ then $g \in \Pi^1(2)$.*

This is a trivial consequence of Lemma 6.

Now we may prove Lemmas 3 through 7 replacing 2 by 1 and 3. Hence we have proved that F is a fundamental domain for Π_3^3.

THEOREM. *E^3/Π_3^3 is a torus.*

This follows from the fact that F is the fundamental domain and the identifications on the boundary of F.

REFERENCES

1. L. Auslander and L. Markus, *Holonomy of flat affinely connected manifolds*, Ann. of Math. vol. 62 (1955) pp. 139–151.

2. N. H. Kuiper, *Sur les surfaces localement affines. Géométrie Différentielle*, Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 79–87, Paris, Centre National de la Recherche Scientifique, 1953.

YALE UNIVERSITY