# LIMIT-PRESERVING EMBEDDINGS OF PARTIALLY ORDERED SETS IN DIRECTED SETS

J. R. ISBELL AND HERMAN RUBIN[1]

Let $(A, >)$ be a partially ordered set. Then we say [3] that a function $f$ on $A$ *decides for* $S$ (on $A$) if $\{\alpha | f(\alpha) \in S\}$ contains a cofinal residual subset of $A$, i.e., if $\gamma \in A$, there is a $\beta \geqq \gamma$ such that for all $\alpha \geqq \beta$, $f(\alpha) \in S$. Clearly the intersection of any two cofinal residual sets is cofinal residual. Ginsburg [2] showed that any partially ordered set $(A, >)$ can be embedded in an everywhere branching set $(B, \succ)$ in such a manner that the "natural" extension of $f$ to $B$ decides for the same sets as $f$ does. Day [1] has an embedding in a directed set preserving decision, but possibly creating new decisions, even turning a nonconvergent function into a convergent one. We show here that the embedding can be made precise. First, let us notice that

LEMMA. *Let $(A, >)$ and $(B, \succ)$ be partially ordered sets and let $\phi$ map $B$ into $A$. If*

(i) *for every $T$ cofinal residual in $A$, $\phi^{-1} T$ is cofinal residual in $B$, and*

(ii) *for every $U$ cofinal residual in $B$, $\phi U$ is cofinal residual in $A$, then for every $f$ and $S$, $f$ decides for $S$ on $A$ if and only if $f\phi$ decides for $S$ on $B$.*

We may assume (by a trivial preliminary embedding) that the given set $A$ has no least cofinal residual set, i.e., the set of maximal elements is not cofinal. Let $B$ be the set of all pairs $(T, \alpha)$ such that $\alpha \in T$ and $T$ is cofinal residual in $A$. We define $(T_1, \alpha_1) \succ (T_2, \alpha_2)$ if $T_1 \subset T_2$ or $T_1 = T_2$ and $\alpha_1 > \alpha_2$. We define $\phi(T, \alpha) = \alpha$. Thus we see that $(B, \succ)$ is directed and therefore cofinal residual reduces to residual. Now if we replace $(A, \alpha)$ by $\alpha$, we see that

THEOREM. *Every partially ordered set $(A, >)$ can be embedded in a directed set $(B, \succ)$ in such a manner that for any function $f$ on $A$, the "natural" extension of $f$ to $B$ decides for precisely those sets for which $f$ decides.*

## REFERENCES

1. Mahlon Day, *Oriented systems*, Duke Math. J. vol. 11 (1944) pp. 201–212.

Received by the editors August 12, 1954 and, in revised form, September 11, 1955.

[1] Authors now (1) at the Institute for Advanced Study and (2) at the University of Oregon.

2. Seymour Ginsburg, *Real-valued functions on partially ordered sets*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 356–359.
3. John Tukey, *Convergence and uniformity in topology*, Princeton, 1940.

PRINCETON UNIVERSITY AND
STANFORD UNIVERSITY

---

# THE ZEROS OF CERTAIN SINE-LIKE INTEGRALS

A. E. LIVINGSTON AND LEE LORCH

We establish here the monotonic character of the zeros (modulo 1) of

$$\int_x^\infty \frac{f(t)}{t}\,dt, \qquad x > 0, \tag{1}$$

where $f(t)$ satisfies the conditions

(C1). $f(t) \geqq 0$ for $0 \leqq t < 1$;

(C2). $f(t) \not\equiv 0$ on any subinterval of $0 \leqq t < 1$;

(C3). $f(t+n) = (-1)^n f(t)$ for $n = 1, 2, 3, \cdots$;

(C4). $f(t)/t$ is Lebesgue integrable on $0 \leqq t \leqq 1$.

It is clear that these conditions imply that the integral (1) has precisely one zero, say $z_n$, in the interval $n < x < n+1$.

Let $C$ be defined (uniquely) by the conditions

$$2\int_0^C f(t)dt = \int_0^1 f(t)dt, \qquad 0 < C < 1. \tag{2}$$

Now,

(A) $z_n - n \geqq C$ for $n = 0, 1, 2, \cdots$,

(B) $z_n - n \to C$ as $n \to \infty$,

as was shown in [2], even more generally, with the factor $1/t$ of $f(t)$ in (1) replaced by a function denoted there by $g(t)$ of which $1/t$ is a special case. When $f(t) = \sin \pi t$, the sequence $\{z_n - n\}_0^\infty$ is decreasing, as Harry Pollard has shown, and I. I. Hirschman has observed that Pollard's proof applies equally well to the zeros of

$$\int_x^\infty g(t) \sin \pi t dt \tag{3}$$

Presented to the Society June 18, 1955; received by the editors December 13, 1954 and, in revised form, October 11, 1955.