A NOTE ON NONRECURRENT RANDOM WALKS[1]

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Let $\{X_i\}$, $i=1, \cdots$, be a sequence of independent and identically distributed random variables with density function $f(x)$. We shall assume throughout that $EX_i=\mu>0$. Let $\{S_n\}$, $n=1, \cdots$, be the sequence of cumulative sums $S_n=\sum_{i=1}^{n} X_i$. Let $H(x)=\sum_{n=1}^{\infty} P(S_n \leqq x)$ and $h(x)=H'(x)$. Since $H(x)$ is nondecreasing, $h(x)$ is defined for almost all x. Let A be any Borel set of positive real numbers and $m(A)$ denote its Lebesgue measure. Chung and Derman [2] proved that

$$P(S_n \in A \text{ infinitely often (i.o.)}) = 0, \qquad m(A) < \infty, \tag{1}$$

and

$$P(S_n \in A \text{ i.o.}) = 1, \qquad m(A) = \infty, \tag{2}$$

if

$$0 < \liminf_{x\to\infty} h(x) \leqq \limsup_{x\to\infty} h(x) < \infty, \tag{3}$$

where only those values of x are considered for which $h(x)$ is defined. (It was also assumed in [2] that $\lim_{x\to-\infty} h(x)=0$. However this assumption can be easily removed.) In this note we shall prove the following

THEOREM 1. (i) *If* $\mu<\infty$ *and* $\limsup_{x\to\infty} h(x)<\infty$, *then*

$$\liminf_{x\to\infty} h(x) > 0;$$

and consequently (1) *and* (2) *hold.*

(ii) *If* $\liminf_{x\to\infty} h(x)>0$, *then* (1) *holds.*

(iii) *If* $\mu<\infty$, *and if there exists a constant* $\alpha>0$ *and an interval* (a, b) *such that* $f(x)\geqq\alpha$ *for* $x\in(a, b)$, *then* $\liminf_{x\to\infty} h(x)>0$.

PROOF OF (i). Since $f(x)$ is a density function there exists an $\epsilon>0$ such that if $E=\{x:f(x)\geqq\epsilon\}$, $m(E)>0$. Since $m(E)>0$, there exists

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a point $x_0 \in E$ such that for $d>0$

$$m(E \cap I_d) = d + o(d) \tag{4}$$

where $I_d = (x_0 - d/2,\ x_0 + d/2)$. Also, since $\limsup_{x\to\infty} h(x) < \infty$, there exist numbers K and x' such that $h(x) < K$ for all $x > x'$. Let d be small enough such that $d - m(E \cap I_d) < d/2K\mu$. This is possible by virtue of (4). It is easily verified that $h(x)$ satisfies the integral equation

$$h(x) = f(x) + \int_{-\infty}^{\infty} f(x-z)h(z)dz. \tag{5}$$

Therefore if $x > x' + x_0 + d/2$

$$\begin{aligned}
h(x) &\geqq \int_{x-z\in I_d} f(x-z)h(z)dz \\
&= \epsilon \int_{x-z\in I_d} h(z)dz + \int_{x-z\in I_d} (f(x-z) - \epsilon)h(z)dz \\
&= \epsilon(H(x - x_0 + d/2) - H(x - x_0 - d/2)) \\
&\quad + \int_{x-z\in I_d\cap E} (f(x-z) - \epsilon)h(z)dz \\
&\quad + \int_{x-z\in I_d\cap \bar{E}} (f(x-z) - \epsilon)h(z)dz \\
&\geqq \epsilon(H(x - x_0 + d/2) - H(x - x_0 - d/2)) - \epsilon K m(I_d \cap \bar{E}) \\
&\geqq \epsilon(H(x - x_0 + d/2) - H(x - x_0 - d/2)) - d\epsilon/2\mu.
\end{aligned}$$

By a result due to Blackwell [1], the first term on the right tends to $d\epsilon/\mu$ as $x \to \infty$. This proves part (i).

Proof of (ii). We shall apply an ergodic theorem. Let Ω be a measurable space with σ-finite measure ν. Let T be a measure preserving transformation of Ω into itself. If Z and Y are two measurable functions defined over Ω, define sequences $\{Z_i\}$ and $\{Y_i\}$, $i=0, 1, \cdots$, by $Z_i(w) = Z(T^i w)$ and $Y_i(w) = Y(T^i w)$ for every $w \in \Omega$. A set Λ is called invariant if $T\Lambda = \Lambda$ a.e. (ν). Let C denote the class of invariant sets. We state part of the ergodic theorem: Suppose $Y > 0$ and $\sum_{i=0}^{\infty} Y_i = \infty$ a.e. (ν). If $\int_\Omega Z d\nu < \infty$ and $\int_\Omega Y d\nu < \infty$, then

$$\lim_{n\to\infty} \left(\sum_{i=0}^{n} Z_i \Big/ \sum_{i=0}^{n} Y_i \right) = U \qquad \text{a.e. } (\nu) \tag{6}$$

and

$$\int_\Lambda UYd\nu = \int_\Lambda Zd\nu \tag{7}$$

for all $\Lambda \in C$.

The proof of this theorem appears in Loeve [5] for the case where ν is a probability measure. However, all the details of the proof carry over for σ-finite measures.

Now consider the Markov chain $\{S_n\}$, $n=0, 1, \cdots$, where $S_n = S_0 + \sum_{i=1}^n X_i$, and $S_0 = s_0$ with probability 1. Let Ω consist of all the sample sequences $w = \{s_0, s_1, \cdots\}$ of $\{S_n\}$. Let ν be the measure, invariant with respect to the shift transformation, constructed by Harris and Robbins [3]. That is, if W is any set in the Borel field obtained by extending the finite dimensional cylinder sets of Ω, then

$$\nu(W) = \int_{-\infty}^{\infty} P(W \mid S_0 = x)dm$$

where $P(\cdot \mid \cdot)$ denotes the conditional probability associated with the Markov chain $\{S_n\}$.

A consequence of the ergodic theorem is

THEOREM 2. *Let* $B = \{x : P(S_n \in A \text{ i.o.} \mid S_0 = x) = 0\}$. *If* $m(A) < \infty$, *then* $m(B) > 0$.

PROOF. Let $B' = \{x : P(S_n \in A \text{ i.o.} \mid S_0 = x) < 1\}$. Suppose $m(B') = 0$. Define

$$\begin{aligned} Y &= 1, & s_0 &\in A, \\ Y &= e^{-|s_0|}, & s_0 &\notin A. \end{aligned}$$

Let $Y_i(w) = Y(T^i w)$, $i = 0, 1, \cdots$, where T is the shift transformation. Then, since $m(B') = 0$, $\nu\{w : \sum_{i=0}^\infty Y_i < \infty\} = 0$. Also, since $m(A) < \infty$, $\int_\Omega Y d\nu < \infty$. Define

$$\begin{aligned} Z &= 1, & s_0 &\in (0, 1), \\ Z &= 0, & s_0 &\notin (0, 1). \end{aligned}$$

Let $Z_i(w) = Z(T^i w)$, $i = 0, 1, \cdots$. The ergodic theorem is applicable. However since $\int_\Omega Z d\nu = 1$, it follows from (7) that there exists some $\Lambda \in C$ such that $\nu(\Lambda) > 0$ and $U(w) > 0$ for all $w \in \Lambda$. Then since $\sum_{i=0}^\infty Y_i = \infty$ a.e. (ν), it follows from (6) that $\sum_{i=0}^\infty Z_i(w) = \infty$ for all $w \in \Lambda$. But by the strong law of large numbers it is easily seen that $P(\sum_{i=0}^\infty Z_i = \infty \mid S_0 = s_0) = 0$ for all s_0. Hence $\nu\{w : \sum_{i=0}^\infty Z_i = \infty\} = 0$. This is a contradiction and therefore the assumption that $m(B') = 0$ is false. A result of Hewitt and Savage [4] implies that $B' = B$. Hence $m(B) = m(B') > 0$. This proves the theorem.

For the remainder of this section A and B will be those sets in Theorem 2.

COROLLARY 1. *For every* x, $m(B\cap(x, \infty))>0$.

COROLLARY 2. *Let* $S_0=0$ *with probability* 1. *If* $P(S_n\in B$ *for some* $n\geqq 1)>0$, *then* $P(S_n\in A$ *i.o.*$)=0$.

We shall omit the proofs of the above corollaries. The proof of Corollary 1 depends on the fact that with probability 1 $S_n>x$ for some n, no matter what the value of x. Under the hypothesis of Corollary 2 it is easy to show that $P(S_n\in A$ i.o.$)<1$. The result of Hewitt and Savage [4] is then applicable.

LEMMA 1. *Let* $S_0=0$ *with probability* 1. *If* $\liminf_{x\to\infty} h(x)>0$, *then* $P(S_n\in B$ *for some* $n\geqq 1)>0$.

PROOF. Since $\liminf_{x\to\infty} h(x)>0$, there exist numbers x' and $\epsilon>0$ such that for all $x>x'$, $h(x)\geqq\epsilon$. Therefore by Corollary 1

$$\int_B h(x)dx \geqq \int_{B\cap(x',\infty)} h(x)dx \geqq \epsilon m(B\cap(x',\infty)) > 0. \tag{8}$$

But the left side of (8) is the expectation of the number of n such that $S_n\in B$. Since it is positive it follows that $P(S_n\in B$ for some $n\geqq 1)>0$. This proves the lemma.

Part (ii) follows from Lemma 1 and Corollary 2.

PROOF OF (iii). From (5) we have

$$\begin{aligned} h(x) &\geqq \int_{x-b}^{x-a} f(x-z)h(z)dz \geqq \alpha\int_{x-b}^{x-a} h(z)dz \\ &= \alpha(H(x-a)-H(x-b)). \end{aligned} \tag{9}$$

By the result of Blackwell [1] the right side of (9) tends to $\alpha(b-a)/\mu>0$ as $x\to\infty$. Therefore part (iii) is proved.

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