UNIQUENESS PROPERTIES OF SETS WITH BLOCKS[1]

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1. Introduction. We show in this paper that any set containing appropriately long blocks of consecutive positive integers is a set of uniqueness for certain significant classes of entire functions of exponential type. We study the classes $K(a, c)$ of entire functions $f(z)$ satisfying

$$f(z) = O(1) \exp (a|x| + c|y| + \epsilon|z|) \text{ for each } \epsilon > 0. \tag{1}$$

If $f(z)$ satisfies (1) but meets the weaker restriction that $f(z)$ is regular for $\mathfrak{R}(z) \geqq 0$, we shall say that it belongs to $K^*(a, c)$. A set A is said to be a set of uniqueness for a class K of functions provided that the only function $f(z)$ in K such that $f(\alpha)=0$ for each α in A is the null function $f(z)=0$. Finally, we say that A has blocks of ratio ϕ if there is an increasing sequence $[n_k]_{k=0,1,2,\cdots}$ such that if $n_k \leqq n \leqq \phi n_k$, then n is contained in A. (If A has blocks of ratio ϕ for each positive ϕ, we say that A has blocks of ratio ∞.) It is our main purpose to prove the following result.

THEOREM 1. *Given $a<\infty$ and $c<\pi$, there is a constant $\phi_0=\phi_0(a, c)$, depending only on a and c, such that every set A with blocks of ratio $\phi>\phi_0$ is a set of uniqueness for $K(a, c)$.*

The function $\sin \pi z$ explains the significance of the condition $c<\pi$, since $\sin \pi z$ belongs to $K(0, \pi)$, yet vanishes for each positive integer n. The minimal character of $\sin \pi z$ is indicated by a celebrated theorem of F. Carlson [1, p. 153],[2] which states that the set I of all positive integers is a uniqueness set for each $K(a, c)$ with $a<\infty$, $c<\pi$. The present work is an extension of Carlson's theorem. It is notable that although Carlson's theorem applies equally well to the larger classes $K^*(a, c)$, our Theorem 1 does not. This is shown by the following theorem.

THEOREM 2. *There is a fixed set A, independent of a and c, and hav-*

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[2] Numbers in brackets refer to references at the end of this paper.

ing blocks of ratio $\phi=\infty$, which fails to be a set of uniqueness for $K^*(a, c)$ whatever the numbers $a<\infty$ and $c>0$.

2. **Proof of Theorem** 1. We shall pretend that we have already found ϕ_0 and will proceed with the proof, in the course of which we shall see how ϕ_0 may be determined. Suppose that the theorem is false. Then there exists a set A with blocks of ratio $\phi>\phi_0$ that is not a set of uniqueness for $K(a, c)$. Thus, there is an increasing sequence $[n_k]$ of positive integers, and a function $f(z)\not\equiv 0$ belonging to $K(a, c)$, such that $f(n)=0$ for each n satisfying $n_k\leqq n\leqq \phi n_k$, $k=0, 1, 2, \cdots$. We shall derive a contradiction, thus proving the theorem.

Let $g(z)=\sum_{n=0}^{\infty} f(n)z^n$. Buck [2] has shown that all of the singular points of $g(z)$, finite and infinite, are contained in the annular sector D whose boundary consists of the concentric circular arcs $|z|=e^a$ and $|z|=e^{-a}$ for $|\arg z|\leqq c$, and the radial segments joining them. Put $s_k(z)=\sum_{n=0}^{k} f(n)z^n$ and $r_k(z)=g(z)-s_k(z)$. Let Γ_1 be the circle $|z|=R_1<e^{-a}$ and let Γ_2 be the contour composed of the circle $|z|=R_2>e^a$ and some closed curve C which encloses D and which is contained in the annular region $R_1<|z|<R_2$. Let ψ be the region whose boundary is formed by Γ_1 and Γ_2.

We estimate $|r_{n_k}(z)|$ on Γ_1, making use of the long block of zero coefficients, writing

$$|r_{n_k}(z)| = \sum_{n\geqq n_k} f(n)z^n = \sum_{n\geqq \phi n_k} f(n)z^n.$$

Hence, for some $\delta_1>0$, we have

$$\log |r_{n_k}(z)| \leqq -\delta_1\phi n_k \text{ for } z \text{ on } \Gamma_1.$$

On Γ_2, $g(z)$ is bounded, and $s_{n_k}(z)$ is a polynomial of degree at most n_k, so that for some δ_2, we have

$$\log |r_{n_k}(z)| \leqq \delta_2 n_k \text{ for } z \text{ on } \Gamma_2.$$

We now choose a convenient circle Γ_3, given by $|z|=R_3$, with $R_2>R_3>e^a$, which lies in ψ and encloses C. We will use the methods of harmonic measure to estimate $\log |r_{n_k}(z)|$ for z on Γ_3. Let $v_1(z)$ and $v_2(z)=1-v_1(z)$ be the harmonic measures in ψ of Γ_1 and Γ_2 respectively. That is, $v_1(z)$ and $v_2(z)$ are harmonic in ψ, continuous in the closure of ψ, and

$$\left.\begin{array}{l} v_1(z) = 1 \\ v_2(z) = 0 \end{array}\right\} z \text{ on } \Gamma_1, \qquad \left.\begin{array}{l} v_1(z) = 0 \\ v_2(z) = 1 \end{array}\right\} z \text{ on } \Gamma_2.$$

By the familiar two constant theorem, we have, for all z in ψ,

$$\log |r_{n_k}(z)| \leqq -\delta_1\phi n_k v_1(z) + \delta_2 n_k v_2(z) = n_k(-\phi\delta_1 v_1(z) + \delta_2 v_2(z)).$$

The choice

$$\phi_0 = (\delta_2/\delta_1) \max_{z\in\Gamma_3} v_2(z)/v_1(z)$$

is now apparent. We have

$$\log |r_{n_k}(z)| \leqq \delta_1 n_k(-\phi + \phi_0) \text{ for } z \text{ on } \Gamma_3,$$

so that $\log |r_{n_k}(z)|$ approaches $-\infty$ uniformly for z on Γ_3. Hence $\lim_{k\to\infty} s_{n_k}(z) = g(z)$ uniformly for z on Γ_3, and therefore for all z within Γ_3. Thus, $g(z)$ is regular throughout the interior of Γ_3. But we have so chosen Γ_3 that it encloses all the singularities of $g(z)$. We conclude, then, that $g(z)$ has no singular points whatever. Hence, $g(z)$ is a constant and we see that $f(n) = 0$ for $n = 1, 2, 3, \cdots$. It now follows from Carlson's theorem that $f(z) \equiv 0$, contradicting our hypothesis, thereby completing our proof.

If we make specific choices of Γ_1, Γ_2, and Γ_3 and determine v_1 and v_2, then an explicit estimate of ϕ_0 may be found. A different method for estimating ϕ_0, by means of Carleman's theorem, is essentially outlined in [**3**]. It leads more easily to an explicit estimate than the present method, but the easier estimate is not as sharp (at least in some special cases) as the present one.

3. **Proof of Theorem** 2. We now construct a set A with blocks of ratio $\phi = \infty$ which is a set of uniqueness for $K^*(a, c)$ for no choice of $a < \infty$, $c > 0$. Let A be the union of the blocks of consecutive integers $n = 2^{2^k}, 2^{2^k}+1, 2^{2^k}+2, \cdots, 2^k 2^{2^k}$ for $k = 1, 2, 3, \cdots$. Clearly, A has blocks of ratio ∞. By a theorem of Fuchs [**1**, p. 157], we shall have proved our theorem if we can show that

$$\limsup_{r\to\infty} [\beta(r) - \epsilon \log r] < \infty \tag{2}$$

for each $\epsilon > 0$, where $\beta(r) = \sum_{n\leqq r} 1/n$, the sum extending only over those n contained in A. For any large r, we may choose k so that $2^{2^k} \leqq r \leqq 2^{2^{k+1}}$. Then

$$\beta(r) \leqq \sum_{v=0}^{k} \sum_{n=2^{2^v}}^{2^v 2^{2^v}} 1/n \sim \sum_{v=0}^{k} \log 2^v = O(k^2).$$

But $\log r \geqq \log 2^{2^k}$. Hence $\beta(r) - \epsilon \log r \leqq O(k^2) - 2^k\epsilon \log 2$, so that $\beta(r) - \epsilon \log r \to -\infty$ as $r \to \infty$, whatever our choice of $\epsilon > 0$, so that (2) is certainly valid. This concludes the proof of the theorem.

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ON COMMUTATORS AND JACOBI MATRICES[1]

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1. All operators in this paper are bounded linear transformations on a Hilbert space. The commutator C of two operators A and B is defined by

$$C = AB - BA. \tag{1}$$

The closure, W, of the set of values (Cx, x) when $\|x\| = 1$ is a closed convex set (Hausdorff, cf. [8, p. 34]). A complex number z will be said to belong to the interior of W if z is in W and if one of the following conditions holds: (i) If W is two-dimensional, then z does not lie on the boundary of W; (ii) If W is a line segment, then z is not an end point; (iii) W consists of z alone.

It was shown in [4] that if A (or B) is normal, or even semi-normal, so that $AA^* - A^*A$ is semi-definite, then 0 belongs to W, but is not necessarily in the interior of W. (That, for arbitrary A and B, in general 0 need not even belong to W was shown in [2].) In fact, if $A = (a_{ij})$ is defined by $a_{i,i+1} = 1$, $a_{ij} = 0$ if $j \neq i+1$, then $C = AA^* - A^*A = (c_{ij})$ is the self-adjoint matrix all elements of which are zero except $c_{11} = 1$. Consequently, $C \geqq 0$ with a spectrum consisting of $\lambda = 0, 1$; hence W is the segment $0 \leqq \lambda \leqq 1$ and 0 is not in the interior of W. Moreover, the above C can also be expressed by $C = DA^* - A^*D$ where D is the (self-adjoint) Jacobi matrix $A + A^*$; cf. § 5 below.

The problem to be considered in the present paper is that of determining a sufficient condition guaranteeing that 0 belongs to the interior of W. Such a condition yields information concerning the spec-

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