PROOF. The uniqueness of M implies that E', the set of idempotents in K, consists of a single point. Therefore K is a group.

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LIMITING SETS OF TRAJECTORIES OF A PENDULUM-TYPE SYSTEM

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We consider the system of differential equations

$$\dot{\theta} = z,\ \dot{z} = G(\theta, z); \tag{1}$$

here $G(\theta, z)$ is continuous in (θ, z) and satisfies some condition (such as a Lipschitz condition) which insures the uniqueness of the solution of (1) at each ordinary point. We also suppose that $G(\theta+2\pi, z) = G(\theta, z)$ for all θ, and that $G(\theta, 0)=0$ has at most a finite number of roots on $0 \leqq \theta < 2\pi$.

It is known (cf. [1, p. 287]) that the solutions of such a system can be studied by means of a cylindrical phase space; i.e., in terms of the curves of the solutions $(\theta(t), z(t))$, the so-called phase trajectories, on the cylinder $r=1$, where (r, θ, z) are cylindrical coordinates. We shall refer to this cylinder $r=1$ as the phase cylinder.

From (1) we obtain the equation

$$z'(\theta) = G(\theta, z)/z(\theta). \tag{2}$$

Suppose $z_1(\theta)$ and $z_2(\theta)$ are solutions of equation (2) such that for $0 \leqq \theta < 2\pi$, $z_1(\theta) > 0$, $z_2(\theta) < 0$, and $z_i(2\pi) < z_i(0)$ for $i=1, 2$, and consider the region R_{12} of the phase cylinder bounded above by the arc of $z_1(\theta)$ for $0 \leqq \theta < 2\pi$ and the line segment joining $z_1(0)$ and $z_1(2\pi)$, and below by the arc of $z_2(\theta)$ for $0 \leqq \theta < 2\pi$ and the line segment joining $z_2(0)$ and $z_2(2\pi)$. We observe that R_{12} must contain the positive

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limiting set (cf. [2, pp. 169–170]) of any trajectory which contains a point of R_{12}, since any such trajectory cannot cross the boundary of R_{12}.

As in the phase plane, positive limiting sets included in a finite region of the phase cylinder must be either (i) critical points, which in this case are the points $(\theta_i, 0)$ where θ_i are the roots of $G(\theta, 0)=0$, and $0 \leqq \theta_i < 2\pi$, or (ii) closed curves which are either single trajectories, so-called cycles, or which consist of trajectories and critical points. On the phase cylinder, however, two distinct types of such closed curves are possible: those which surround the cylinder, and those which do not. We shall refer to closed curves which surround the cylinder as of L_2 type, the other kind as of ordinary type.

THEOREM. *If* $\lim_{|z|\to\infty} G(\theta, z)/z = H(\theta)$ *uniformly for all* θ *and if* $\int_0^{2\pi} H(\theta)d\theta < 0$, *then the positive limiting set of the trajectory of each solution of* (1) *is contained in a finite part of the phase cylinder and consists of one of the following configurations*:

(i) *a critical point of* (1),

(ii) *a closed curve of* L_2 *type which may be a* L_2 *cycle*,

(iii) *a closed curve of ordinary type which may be an ordinary cycle.*

From the discussion preceding the statement of the theorem, we observe that this theorem is an immediate consequence of the following:

LEMMA. *Under the assumptions of the preceding theorem, there exists a constant* $M>0$ *such that for each solution* $z(\theta)$ *of equation* (2) *for which* $|z(0)| > M$, *we have* $z(\theta) \neq 0$ *for* $0 \leqq \theta < 2\pi$, *and* $z(0) > z(2\pi)$.

PROOF. Let ϵ be fixed such that $0 < 2\pi\epsilon < K(2\pi)$, where

$$K(\theta) = -\int_0^\theta H(s)ds.$$

Since $H(\theta)$ is the uniform limit of continuous functions, the above integral exists, as does a number $M(\epsilon)$ such that for all θ, $|(G(\theta, z)/z) - H(\theta)| < \epsilon$ whenever $|z| > M(\epsilon)$. Define $M = \max\ (2\pi\epsilon + m,\ M(\epsilon))$ where

$$m = \int_0^{2\pi} |H(s)|\, ds,$$

and consider a solution $z(\theta)$ of equation (2) such that $|z(0)| > 2M_0$. We show first that $|z(\theta)| > M_0$ for $0 < \theta < 2\pi$. For suppose at b, $0 < b < 2\pi$, $|z(b)| = M_0$ while $|z(\theta)| > M_0$ for $0 < \theta < b$. We have, clearly, that $|z'(\theta)| \leqq |(G(\theta, z)/z) - H(\theta)| + |H(\theta)|$ from which it follows that $|z'(\theta)| < \epsilon + |H(\theta)|$. Integrating this last inequality from 0 to b we have

$$\int_0^b |z'(\theta)| d\theta < b\epsilon + \int_0^b |H(\theta)| d\theta < 2\pi\epsilon + m \leqq M_0. \tag{3}$$

However,

$$\int_0^b |z'(\theta)| d\theta \geqq \left| \int_0^b z'(\theta) d\theta \right| > M_0; \tag{4}$$

and from (3) and (4) we have a contradiction. Hence, any solution $z(\theta)$ of equation (2) for which $|z(0)| > 2M_0$ must be such that $|z(\theta)| > M_0$ for $0 \leqq \theta < 2\pi$.

We now observe that for such $z(\theta)$ we have

$$z'(\theta) - H(\theta) = G(\theta, z(\theta))/z(\theta) - H(\theta) < \epsilon,$$

and integrating from 0 to 2π, we have $z(2\pi) - z(0) + K(2\pi) < 2\pi\epsilon$, or $z(2\pi) - z(0) < 2\pi\epsilon - K(2\pi) < 0$. If we define $M = 2M_0$, the proof of the lemma is thus complete.

We remark that in the case for which $G(\theta, 0) = 0$ has no roots, the positive limiting sets can only be closed curves of L_2 type. This follows since in this case (1) has no critical points.

In conclusion, we point out that stronger results concerning the nature of these finite positive limiting sets have been obtained [3] for the special case of (1) where $G(\theta, z) = g(\theta) - f(\theta, \alpha)z$. More precisely, conditions involving the dependence of $f(\theta, \alpha)$ on α are known for which each limiting set is a critical point or a closed curve of ordinary type, and under additional conditions, each positive limiting set must be a critical point. These conditions, moreover, do not require that $f(\theta, \alpha) \geqq 0$ for all θ.

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