A THEOREM ON PARALLELS

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For the theorem which we wish to prove we need Hilbert's axioms of incidence I 1–3 referring to plane geometry and the axioms of order [**2**, pp. 3–5]. Special emphasis will be placed on the second axiom of order, II 2: If A and C are two points of a straight line, there exists at least one point B of the line so situated that C lies between A and B. Now the main purpose of our theorem is to show the existence of parallels using continuity and not congruence as is usually the case in plane geometry. Therefore we have to state our axioms of continuity in a form not involving congruence, known as the postulate of Dedekind [**1**, p. 23]: For every partition of all the points of a segment into two nonvacuous sets, such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.

We prove the following

THEOREM. *Given a line q and a point P not on the line in a plane geometry satisfying the axioms of incidence, order, and continuity as stated above, there exists at least one line through P which does not intersect q. If the axioms of incidence and order are retained and Dedekind's axiom is not assumed, all lines may intersect.*

An indirect proof will be used for the first part of our theorem assuming that all lines passing through P intersect q. Because of Axiom I 3 there are two points, A and B, on q. Also, on account of II 2 there exists a point C such that P is between A and C. Consider now a point X on q such that A lies between X and B and a point Y such that B is between Y and A. Using P as center of projection we obtain the projections X' and Y' of these points on line BC. If we take all points X and Y as described, their projections X' and Y' constitute two sets which follow the order relationship required by the postulate of Dedekind. The latter fact can be ascertained by reasoning along the lines of [**2**, pp. 5–8]. Moreover, the points of type X' and Y' exhaust the segment BC since we are assuming that all lines through P intersect q. Hence according to Dedekind's axiom there exists a point producing this division of all points of segment BC into two sets. The projector of this point meets q in a point beyond which there are no further points on q. But there is no last point as we know from

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Axiom II 2. We have arrived at a contradiction thus proving part one of our theorem.

For the second part of our theorem we show by an example that if continuity is not postulated, no parallels need exist. We construct the example in the augmented Euclidean plane in which x_1, x_2, x_3 are homogeneous Cartesian coordinates. Take the set of all points such that x_1, x_2, x_3 can be written as rational numbers. Similarly, take all lines with line coordinates u_1, u_2, u_3 that can be made rational. The incidence relation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

shows that any two distinct points of the set of points determine a line of the set of lines being considered. The axioms of incidence are satisfied. Furthermore two such lines always intersect in one of the rational points.

For the purpose of getting an order relation we introduce a line given by the equation

$$2^{1/2}x_1 + 3^{1/2}x_2 + 5^{1/2}x_3 = 0.$$

It is not one of our rational lines and it is not incident with any of the rational points. The new line is used as an ideal line in the following sense. Let A, B, C be three rational points on a rational line and let I be its intersection with the ideal line. We agree to say that C is between A and B if the pair of points A, B is separated by the pair of points I, C. It is easily seen that this definition of three point order satisfies the axioms of order, especially II 2.

Obviously the postulate of Dedekind is not fulfilled in our system. Since all lines intersect the proof of the theorem is complete.

References

1. H. S. M. Coxeter, *Non-Euclidean geometry*, 2d ed., Toronto, 1947.
2. D. Hilbert, *Grundlagen der Geometrie*, 7th ed., Leipzig and Berlin, 1930.

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