3. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, 1935.

4. ———, *The location of critical points of analytic and harmonic functions*, Amer. Math. Soc. Colloquium Publications, vol. 34, 1950.

5. ———, *An interpolation problem for harmonic functions*, Amer. J. Math. vol. 76 (1954) pp. 259–272. Also *Determination d'une fonction analytique par ses valeurs données dans une infinité dénombrable de points*, Bull. Soc. Math. de Belgique (1955) pp. 52–70.

6. J. L. Walsh and Philip Davis, *Interpolation and orthonormal systems*, Journal d'Analyse Mathématique vol. 2 (1952) pp. 1–28.

HARVARD UNIVERSITY AND
SMITH COLLEGE

PARTIALLY BOUNDED CONTINUED FRACTIONS

H. S. WALL

For each complex number sequence a, $f(a)$ denotes the continued fraction

$$\frac{1}{1}+\frac{a_1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\cdots.$$

The statement that $f(a)$ is *partially bounded*[1] means that the sequence a has a bounded infinite subsequence. If $f(a)$ is partially bounded, the series $\sum|b_p|$ diverges, where $b_1=1$, $a_p=1/b_pb_{p+1}$, $p=1, 2, \cdots$, —a necessary condition for convergence of $f(a)$.

Any continued fraction $f(a)$ such that $\sum|b_p|$ diverges is convergent provided its even and odd parts are absolutely convergent,[2] i.e. provided the series $\sum|f_{2p+2}-f_{2p}|$ and $\sum|f_{2p+1}-f_{2p-1}|$ are convergent, where $\{f_p\}_{p=1}^{\infty}$ is the sequence of approximants. The *simple* convergence of the even and odd parts of $f(a)$, together with the divergence of $\sum|b_p|$, is *not* sufficient for the convergence of $f(a)$, (Theorem 3). However, the simple convergence of the even and odd parts of the partially bounded continued fraction $f(a)$ is sufficient for the convergence of $f(a)$. In fact, we have this theorem:

THEOREM 1. *Suppose there is a positive integer k such that the sub-*

Presented to the Society, December 27, 1955; received by the editors October 27, 1955 and, in revised form, December 27, 1955.

[1] $f(a)$ is called *bounded* if the sequence a is bounded—a condition equivalent to the boundedness of a certain infinite matrix. Cf. H. S. Wall, *Analytic theory of continued fractions*, 1948, p. 110. (Referred to later on as AT.)

[2] Trans. Amer. Math. Soc. vol. 67 (1949) pp. 368–380.

sequence $\{f_p\}_{p=k}^{\infty}$ *of the sequence of approximants of the partially bounded continued fraction* $f(a)$ *is bounded. If the even* (*odd*) *part of* $f(a)$ *converges and has the value* v, *then there is an infinite subsequence of the sequence of approximants of the odd* (*even*) *part of* $f(a)$ *which converges to* v.

PROOF. Let A_p and B_p be the pth numerator and denominator of $f(a)$, so that $A_0=0$, $A_1=1$, $B_0=1$, $B_1=1$,

$$A_{p+1} = A_p + a_pA_{p-1}, \quad B_{p+1} = B_p + a_pB_{p-1}, \qquad p = 1, 2, \cdots,$$

and

$$A_pB_{p+1} - A_{p+1}B_p = A_pB_{p+2} - A_{p+2}B_p = (-1)^{p+1}a_1a_2 \cdots a_p, \qquad p = 1, 2, \cdots.$$

Therefore, if $p \geqq k$,

$$f_p - f_{p+1} = \frac{(-1)^{p+1}a_1a_2 \cdots a_p}{B_pB_{p+1}}, \quad f_p - f_{p+2} = \frac{(-1)^{p+1}a_1a_2 \cdots a_p}{B_pB_{p+2}},$$

and consequently[3]

$$-a_{p+2}(f_p - f_{p+2})(f_{p+1} - f_{p+3}) = (f_p - f_{p+1})(f_{p+2} - f_{p+3}),$$

or

$$| a_{p+2}(f_p - f_{p+2})(f_{p+1} - f_{p+3}) | = | (f_p - f_{p+1})(f_{p+2} - f_{p+3}) |.$$

Suppose the even (odd) part of $f(a)$ converges to v and that no infinite subsequence of the sequence of approximants of the odd (even) part of $f(a)$ converges to v. Then, there exists a positive number c and a positive integer m greater than k such that if p is a positive integer greater than m, $|(f_p-f_{p+1})(f_{p+2}-f_{p+3})| \geqq c$. Let M be a number such that, if $p \geqq k$, $|f_p-f_{p+2}| \leqq M$, and let L be a number such that, for each positive integer r there exists a positive integer s greater than r for which $|a_{s+2}| \leqq L$. There exists a positive integer N greater than m such that if p is an integer greater than N, $|f_p-f_{p+2}| < c/ML$ or $|f_{p+1}-f_{p+3}| < c/ML$. Hence, there exists a positive integer s greater than N such that $c \leqq |a_{s+2}(f_s-f_{s+2})(f_{s+1}-f_{s+3})| \leqq LM \cdot (c/LM) = c$. This contradiction proves our supposition false and establishes the theorem.

[3] This formula expresses the fact that a_{p+2} is a cross-ratio of four successive approximants of $f(a)$. Cf. Wall, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 578–592.

THEOREM 2. *If the even and odd parts of the partially bounded continued fraction $f(a)$ converge, then $f(a)$ converges.*

This is an immediate corollary to Theorem 1.

THEOREM 3. *There exists a divergent continued fraction $f(a)$ whose even and odd parts converge for which the series $\sum |b_p|$ diverges.*

PROOF. Let a denote the sequence defined by $a_p = (-1)^p(p+1)^2$, $p=1, 2, \cdots$. Since $|b_p b_{p+1}|^{1/2} = 1/(p+1) \leqq [|b_p| + |b_{p+1}|]/2$, the series $\sum |b_p|$ diverges. The even part of $f(a)$ is $-f(c)/3$, where c is the sequence defined by $c_p = p^2(2p+1)/(2p-1)$, $p=1, 2, \cdots$, so that the even part of $f(a)$ converges[4] and its value is a negative number. The odd part of $f(a)$ is $1+2f(d)/3$, where d is the sequence defined by $d_p = (p+1)^2(2p+1)/(2p+3)$, $p=1, 2, \cdots$, so that the odd part of $f(a)$ converges and its value is a number greater than 1. Hence $f(a)$ diverges.

A corollary to Theorem 2 is the following theorem of Farinha.[5] Let E denote a bounded closed set in the complex plane which does not contain 0. If the even and odd parts of $f(a)$ converge uniformly for a_i in E, then $f(a)$ converges uniformly for a_i in E.

As an application of Theorem 2, we shall prove the following extension of a theorem of Thron.[6]

THEOREM 4. *Suppose r is a positive number not greater than 1 and s a positive number less than $(1+r)^{-2}$. The continued fraction $f(a)$ such that $|a_{2p}| \leqq r^2$ and $|1/a_{2p-1}| \leqq (1+r)^{-2} - s$, $p=1, 2, \cdots$, is convergent.*

PROOF. Since $f(a)$ is partially bounded, it suffices to show that its even and odd parts converge.

The even part of $f(a)$ may be written as

$$\frac{i/a_1}{b_1 + is} - \frac{c_1^2}{b_2 + is} - \frac{c_2^2}{b_3 + is} - \cdots,$$

where

$$b_p = i + \frac{i}{a_{2p-1}} + \frac{a_{2p-2}}{a_{2p-1}} - is, \quad c_p^2 = -\frac{a_{2p}}{a_{2p+1}}, \quad p = 1, 2, \cdots, (a_0 = 0).$$

[4] AT, pp. 58–59.

[5] Rev. Fac. Ci. Univ. Coimbra vol. 23 (1954) pp. 17–20.

[6] Duke Math. J. vol. 10 (1943) pp. 677–685, p. 680. Thron restricts r to be less than 1. Cf. Leighton and Wall, Amer. J. Math. vol. 58 (1936) pp. 267–281, p. 267, for the first theorem of this type.

For each positive integer p, $\beta_p = \operatorname{Im} b_p \geqq 2r(1+r)^{-2} + r^2 s > 0$, and there exists a number θ_p and a number t_p such that

$$|c_p^2| = \left(\frac{r}{1+r}\right)^2 \theta_p, \qquad \operatorname{Re} c_p^2 = \left(\frac{r}{1+r}\right)^2 \theta_p t_p,$$
$$0 \leqq \theta_p \leqq 1, \ -1 \leqq t_p \leqq 1.$$

Then,

$$0 \leqq \frac{|c_p^2| - \operatorname{Re} c_p^2}{2\beta_p \beta_{p+1}} \leqq \frac{(1+r)^2 \theta_p (1 - t_p)}{2(2 + r - r\theta_p t_p)(2 + r - r\theta_{p-1} t_{p-1})}.$$

Since the derivative with respect to r of the last expression is non-negative, it does not exceed

$$\frac{2\theta_p(1 - t_p)}{(3 - \theta_p t_p)(3 - \theta_{p-1} t_{p-1})},$$

so that, if $h_p = (1 - t_p)/2$ and $g_p = [1 + \theta_p h_p]^{-1}$, we have:

$$|c_p|^2 - \operatorname{Re} c_p^2 \leqq 2\beta_p \beta_{p+1}(1 - g_p) g_{p+1},$$
$$0 < g_p \leqq 1, \ p = 1, 2, \cdots.$$

Thus, the even part of $f(a)$ is positive definite[7] and, being bounded, is convergent.[8]

The odd part of $f(a)$ may be proved convergent in the same way, completing the proof of Theorem 4.

George Copp[9] has improved Theorem 4 by showing that $f(a)$ converges if there is a positive number r *not greater than* 1 such that $|a_{2p}| \leqq r^2$ and $|a_{2p-1}| \geqq (1+r)^2$, $p = 1, 2, \cdots$. He also showed that $f(a)$ converges if there exists a positive number r less than 1 such that, for each positive integer p, $|a_{2p}| \leqq r^2$ and $|a_{2p-1}| \leqq (1-r)^2$ or $|a_{2p-1}| \geqq (1+r)^2$.

UNIVERSITY OF TEXAS

[7] AT, p. 69.

[8] AT, p. 112.

[9] George Copp, *Some convergence regions for a continued fraction*, Dissertation, The University of Texas, 1950.