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THE EXISTENCE OF OUTER AUTOMORPHISMS OF SOME GROUPS

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F. Haimo and E. Schenkman [1] raised the question: Does a nilpotent group G always possess an outer automorphism? The answer is in the affirmative if G is finite and nilpotent of class 2, as is seen from a Schenkman's [1] stronger result. The object of this note is to show that the answer is also in the affirmative for another family of nilpotent groups, namely the family of all finite p-groups G of order greater than 2 such that $x^p = e$ for every element x in G. Actually, our result is somewhat stronger:

THEOREM 1. *Suppose that G is a group every element of which is of order a divisor of a fixed integer $n > 1$. If G has a normal subgroup N such that the factor group G/N is cyclic of order n and such that the intersection $N \cap Z$ of N with the center Z of G contains an element a_0 of order n, then G possesses an outer automorphism which induces identity automorphisms on both N and G/N.*

For the proof we need the following

LEMMA. *If elements $a, a_0, a_1, \cdots, a_k$ in a group G are such that $aa_0 = a_0 a$, $a_i a_j = a_j a_i$ for $i, j = 0, 1, \cdots, k$ and such that*

$$a_i a a_i^{-1} = a a_{i-1} \qquad (i = 1, 2, \cdots, k) \tag{1}$$

then we have, for $r = 1, 2, \cdots,$

$$(a a_k)^r = a^r a_k^{C_{r,1}} a_{k-1}^{C_{r,2}} \cdots a_{k-r+1}^{C_{r,r}}, \tag{2}$$

where we set $a_{-1} = a_{-2} = \cdots = e$. If, moreover, a_0 is of order n and if $a_i^n = e$ for $i = 1, 2, \cdots, k$, then

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(3) $$a_0^{s_0} a_1^{s_1} \cdots a_k^{s_k} = e$$

implies $s_0 \equiv s_1 \equiv \cdots \equiv s_k \equiv 0 \pmod{n}$.

PROOF. Obviously (2) is true for $r=1$. Suppose that (2) is true for a fixed value of r. Since (1) implies $a^{-1}a_i a = a_i a_{i-1}$, we have

$$\begin{aligned}(aa_k)^{r+1} &= a^r a_k^{C_{r,1}} a_{k-1}^{C_{r,2}} \cdots a_{k-r+1}^{C_{r,r}}\, a\, a_k \\ &= a^{r+1}(a_k a_{k-1})^{C_{r,1}} (a_{k-1}\, a_{k-2})^{C_{r,2}} \cdots (a_{k-r+1}\, a_{k-r})^{C_{r,r}}\, a_k \\ &= a^{r+1} a_k^{C_{r+1,1}} a_{k-1}^{C_{r+1,2}} \cdots a_{k-r}^{C_{r+1,r+1}}.\end{aligned}$$

Therefore (2) is true for all $r=1, 2, \cdots$.

We shall prove the second parts. First we observe that $a_0^{s_0}=e$ implies $s_0\equiv 0 \pmod{n}$, since a_0 is of order n. In order to proceed by induction with respect to k, we assume that the second part of our lemma is true if $s_k=0$. From (3) we have, as before, by transformation with a,

$$a_0^{s_0}(a_0\, a_1)^{s_1} \cdots (a_{k-1}\, a_k)^{s_k} = e.$$

Hence $a_0^{s_1} \cdots a_{k-1}^{s_k}=e$, and hence the inductive assumption yields $s_1 \equiv \cdots \equiv s_k \equiv 0 \pmod{n}$. But then (3) yields $a_0^{s_0}=e$ and $s_0\equiv 0 \pmod{n}$, since $a_i^n=1$ for $i=1, 2, \cdots, k$. Thus the second part is also proved.

PROOF OF THEOREM 1. We shall derive a contradiction by assuming that every automorphism of G which induces identity automorphisms on both N and G/N is inner. Let aN be a generating element of the group G/N. We have already a_0 in the center Z_N of N. Suppose that we have found elements $a_0, a_1, \cdots, a_k$ in Z_N such that (1) holds. We shall show that the mapping σ defined by $(ua^r)^\sigma = u(aa_k)^r$, where $u\in N$, is an automorphism of G. We have $(ua^r va^s)^\sigma = (u[a^r, v]va^{r+s})^\sigma = u[a^r, v]v(aa_k)^{r+s}$, while $(ua^r)^\sigma(va^s)^\sigma = u(aa_k)^r v(aa_k)^s = u[(aa_k)^r, v]v(aa_k)^{r+s}$. Therefore we have to show $[(aa_k)^r, v]=[a^r, v]$. But this follows immediately from the first part of the lemma, since $(aa_k)^r = a^r x$ with an element x in the center of N. Suppose now that $(ua^r)^\sigma = ua^r x = e$. Then $r\equiv 0 \pmod{n}$. Hence $(ua^r)^\sigma = u = ux = e$, and hence $ua^r=e$. We can also see easily that every element in G is an image of the mapping σ. Therefore σ is an automorphism. Since σ induces identity automorphisms on both N and G/N, there exists a_{k+1} in G such that $g^\sigma = a_{k+1} g a_{k+1}^{-1}$ for g in G. We shall show that a_{k+1} is in N. Let $a_{k+1} = ua^r$ with u in N. Then $a_{k+1}^\sigma = a_{k+1}$ implies $(aa_k)^r = a^r$. Then by our lemma $r\equiv 0 \pmod{n}$, and hence $a_{k+1}=u$ is in N. Since a_{k+1} commutes with every element in N, a_{k+1} is in Z_N. Thus we have proved that there exist elements $a_0, a_1, \cdots, a_{n-1}$ in Z_N such that (1) holds with $k=n-1$. Now, by (2), we have $(aa_{n-1})^n=a^na_{n-1}^n \cdots a_1^na_0$. Since $x^n=1$ for every x in G, we have $a_{n-1}^n \cdots a_1^na_0=e$. Then the second part of our lemma implies $1\equiv 0 \pmod n$, a contradiction. Thus the proof is complete.

By a similar method, we can prove the following

THEOREM 2. *Let p be an odd prime, and G a nilpotent group of class less than p. If G has a normal subgroup N of index p such that every element $\neq e$ in the center Z_N of N is of order p, then G possesses an outer automorphism which induces identity automorphisms on both N and G/N.*

PROOF. Let $z\neq e$ be an element in the center of G. Suppose that z is not in N. If $z^p=e$ then $G=(z)\times N$, and hence Theorem 2 is trivial. If $z^p\neq e$ then z is an element $\neq e$ in Z_N. Thus we may assume at the outset that z is in Z_N. We set $a_0=z$, and find $a_1, a_2, \cdots, a_{p-1}$ as in the proof of Theorem 1. This is possible, since $(aa_k)^p=a^p$ for $k=0, 1, \cdots, p-2$, as is seen from (2). Let $G\supset G_1\supset G_2\supset \cdots$ be the lower central series of G. Then (1) shows that a_0 is in $G_{p-1}=(e)$. This is a contradiction derived from the assumption that every automorphism which induces identity automorphisms on both N and G/N is inner. Thus Theorem 2 is proved.

Consider now a torsion free nilpotent group G satisfying maximal condition for subgroups. For such a group G it is known [1] that the factor groups formed by successive groups appearing in the upper central series of G are torsion free. Therefore G has a normal subgroup N containing the center of G such that G/N is infinite cyclic. By applying the above method we obtain

THEOREM 3. *Any torsion free nilpotent group satisfying maximal condition for subgroups has an outer automorphism.*

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