SOME COMBINATORIAL EXTREMUM PROBLEMS

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1. **Introduction.** This paper is concerned with the extremization of a function of finitely or infinitely many variables, into the structure of which neighbor relations enter between the subscripts of the variables given by a linear graph, over all permutations of the variables.

More precisely: let P be the set of vertices of a connected linear graph γ and S a given set of objects of the same cardinality as P. A permutation x of S is a mapping of S onto P, and $F(x)$ assigns a value to each permutation. Our problem is that of finding the extremal values, if any, of $F(x)$ and to determine the x at which they are attained.

In general the solution of this problem will involve an unduly large number of trials for finite graphs and will, for infinite graphs, not be feasible by trial methods. In §2 we introduce a class of functions $F(x)$, namely those satisfying equation (1) for some ordering of S, for which the problem can be solved explicitly.

In §§3 and 4 we apply the method to problems of "shortest route" type [1], and to finding extremal denominators of continued fractions. This latter question was posed by C. A. Nicol; it allows a ready application of the Main Lemma since condition (1) is satisfied for the natural order of the denominators. It was this case which gave rise to the general method. In the case of shortest route problems, condition (1) is equivalent to a four point condition.

Finally, in §5, we state some unsolved problems.

2. **Definitions and Main Lemma.** By a linear graph we mean a graph in which no point is joined to more than two points. A point with only one neighbor is an *end point*. The connected linear graphs with at least two points are the following: *Segment*, finite graph with two end points; *cycle*, finite graph with no end points; *half-line*, infinite graph with one end point; *line*, infinite graph with no end points. Consider a connected linear graph γ and a function $F_\gamma(x_1, x_2, \cdots)$ where x_i is a variable associated to the point P_i of the graph. Let $a_1 \geqq a_2 \geqq \cdots \geqq a_i \geqq \cdots$ be a given set of numbers. We then let $(x_1, x_2, \cdots)$ vary over the permutations of $(a_1, a_2, \cdots)$ and ask for the permutations for which F_γ is extremal.

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MAIN LEMMA. *Let σ be a segment in the connected linear graph γ and let $F_{\gamma,\sigma}(x_1, x_2, \cdots)$ be obtained from $F_\gamma(x_1, x_2, \cdots)$ by reversing the order of the points of σ. Let P_i, P_j be the end points of σ and $P_{i'}$, $P_{j'}$ the neighbors of P_i, P_j in $\gamma-\sigma$ (if P_i or P_j has no neighbor in $\gamma-\sigma$ then we define its neighbor as $P_{-\infty}$ to which we associate the value $-\infty$ with the convention $-\infty-(-\infty)=0$). If, for every permutation and every segment σ, the relation*

$$\text{(1)} \qquad \operatorname{sgn}(F_\gamma(x) - F_{\gamma,\sigma}(x)) = \operatorname{sgn}(x_i - x_j)(x_{i'} - x_{j'})$$

holds, then max $F_\gamma(x)$ *is not attained for half-lines and is otherwise obtained when (x) is the permutation*

$$\text{(2)} \qquad \cdots a_5\ a_3\ a_1\ a_2\ a_4 \cdots$$

where numbers written next to each other correspond to neighboring points and in the cycle case the extremes are neighbors. Under the same condition min $F_\gamma(x)$ *is not attained for infinite graphs, while for a segment it is attained by*

$$\text{(3)} \qquad a_1\ a_n\ a_3\ a_{n-2} \cdots a_4\ a_{n-1}\ a_2$$

and for a cycle by

$$\text{(4)} \qquad \cdots a_3\ a_{n-1}\ a_1\ a_n\ a_2 \cdots$$

where the ends are neighbors.[2]

PROOF. Equation (1) can be interpreted as follows. If $F_\gamma(x)$ is maximal then for every segment σ in γ the greater end point (that is the end point with the greater a_i attached to it) has the greater neighbor in $\gamma-\sigma$. If $F_\gamma(x)$ is minimal then for every σ the greater end point has the smaller neighbor in $\gamma-\sigma$.

For $a_1 > a_2 > \cdots$, this determines the order up to permutations which leave γ invariant. For the maximum arrangement, if a_2 is not a neighbor of a_1, then by inverting a segment we can make it into a neighbor and increase F_γ; similarly a_3 can be made the other neighbor of a_1, etc. to obtain (2). The same reasoning leads to (3) and (4) and shows also that for infinite graphs and any given arrangement there is another with a smaller value of $F_\gamma(x)$; similarly for half-lines and max. For $a_1 \geqq a_2 \geqq \cdots$ the arrangements are still uniquely determined but for the obvious possibility of interchanging equal numbers.

[2] *Added in proof.* The extremality of (2) and (3) under certain conditions was found independently by F. Supnick [4] who earlier had given a special application [3].

We can likewise solve some extremal problems with side conditions. As an example we may consider the case where γ is a segment with values at the end points prescribed as a_i, a_j. Assuming $i<j$ we get the maximal arrangement

$$a_i\ a_{i-2}\ a_{i-4}\cdots a_1\cdots a_{i-3}\ a_{i-1}\ a_{i+1}\ a_{i+2}\cdots a_{j-1}\ a_{j+1}\ a_{j+3}\cdots a_n\cdots a_{j+2}\ a_j;$$

the minimal arrangement is analogously obtained as a modification of (4).

3. **Shortest route problems.** We now restrict our attention to functions F_γ of the form

$$F_\gamma(x) = \sum f(x_i, x_{i'})$$

where the summation is extended over all consecutive pairs of points in the oriented graph γ. If the a_i correspond to points of a set S and $f(a_i, a_j)$ is considered a distance then the extremal problem for $F_\gamma(x)$ is an extremal problem for the length of a route through S.

THEOREM 1. *Let $f(x, y)$ be a symmetric function. Then $F_\gamma(x)$ satisfies condition* (1) *of the Main Lemma if and only if f satisfies the four point condition*

$$\operatorname{sgn}\left(f(x_1, y_1) + f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1)\right) = \operatorname{sgn}(x_1 - x_2)(y_1 - y_2). \tag{5}$$

The proof is obvious.

Note that (5) means that for any four points, (2) and (4) give the maximizing and minimizing arrangement among the three possible cycles.

We observe that (5) is satisfied if

$$\frac{\partial^2 f}{\partial x \partial y} > 0$$

everywhere. Let us therefore consider the example

$$f(x, y) = xy$$

and

$$a_k = \frac{1}{k}\cdot$$

According to §2 we obtain the following:

Segment: $$\max F_\gamma = 1\cdot\frac{1}{2} + \left(1\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{5} + \cdots\right) + \left(\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{6} + \cdots\right)$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4[(n-1)/2]+2} + \frac{1}{4} - \frac{1}{4[n/2]}$$
$$= \frac{5}{4} - \frac{1}{2n} - \frac{1}{2(n-1)},$$
$$\min F_\gamma = 1\cdot\frac{1}{n} + \sum_{k=1}^{[(n-1)/2]} \frac{1}{k(n-k+1)} + \sum_{k=0}^{[(n-3)/2]} \frac{1}{(k+2)(n-k+1)}$$
$$= \frac{2}{n}\left(\log n + \gamma - \frac{3}{4}\right) + O\left(\frac{1}{n^2}\right).$$

Cycle: $$\max F_\gamma = \frac{5}{4} - \frac{1}{2n} - \frac{1}{2(n-1)} + \frac{1}{n(n-1)}$$
$$= \frac{5}{4} - \frac{2n-3}{2n(n-1)},$$
$$\min F_\gamma = 1\cdot\frac{1}{n} + \sum_{k=1}^{[(n-1)/2]} \frac{1}{k(n-k)} + \sum_{k=0}^{[(n-3)/2]} \frac{1}{(k+2)(n-k)}$$
$$= \frac{2}{n}(\log n + \gamma) + O\left(\frac{1}{n^2}\right).$$

Line: $$\max F_\gamma = \frac{5}{4}\cdot$$

It will in general not be possible to number points in the plane so that their distances satisfy condition (5); there are however cases in which we can apply these results. For example let $P_i = (x_i, y_i)$ $(i = 1, \cdots, n)$, $x_1 < x_2 < \cdots < x_n$ be a set of points in the plane so that $|y_{i+1} - y_i| < x_{i+1} - x_i$. We wish to find the closed path joining these points which minimizes the sum of the squares of the distances between consecutive points. We can number the points by x_i and we have

$$f(x, x') = (x - x')^2 + (y(x) - y(x'))^2$$
$$(y(x_i) = y_i \text{ and linear in between})$$

with $\partial^2 f/\partial x \partial x' = -2-2(dy/dx)(x)(dy/dx)(x') < 0$ so that $-f$ satisfies the condition of Theorem 1. The minimizing route is thus $\cdots P_5 P_3 P_1 P_2 P_4 \cdots$ where the end points are neighbors.

4. **Continued fractions.** We now let γ be a segment and $F_\gamma(x)$ the denominator $[x_1, \cdots, x_n]$ of the continued fraction $1/x_1 + \cdots + 1/x_n$.

LEMMA. *If $m = \min x_i > 0$ and $|x_i - x_j| \geqq 1/m$ $(i \neq j)$ then $F_\gamma(x) = [x_1, \cdots, x_n]$ satisfies condition* (1) *of the Main Lemma.*

PROOF.

$$\begin{aligned} F_\gamma(x) &= [x_1, \cdots, x_{k-1}, x_k, \cdots, x_{k+l}, x_{k+l+1}, \cdots, x_n] \\ &= [x_1, \cdots, x_{k-1}][x_k, \cdots, x_{k+l}][x_{k+l+1}, \cdots, x_n] \\ &\quad + [x_1, \cdots, x_{k-2}][x_{k+1} \cdots x_{k+l}][x_{k+l+1} \cdots x_n] \\ &\quad + [x_1 \cdots x_{k-1}][x_k \cdots x_{k+l-1}][x_{k+l+2} \cdots x_n] \\ &\quad + [x_1 \cdots x_{k-2}][x_{k+1} \cdots x_{k+l-1}][x_{k+l+2} \cdots x_n]. \end{aligned}$$

If σ is the segment $x_k \cdots x_{k+l}$ then, since $[c_1 \cdots c_m] = [c_m \cdots c_1]$, we obtain

$$\begin{aligned} &F_\gamma(x) - F_{\gamma,\sigma}(x) \\ &\quad = ([x_1 \cdots x_{k-2}][x_{k+l+1} \cdots x_n] - [x_1 \cdots x_{k-1}][x_{k+l+2} \cdots x_n]) \\ &\qquad \times ([x_{k+1} \cdots x_{k+l}] - [x_k \cdots x_{k+l-1}]) \\ &\quad = \{(x_{k+l-1} - x_{k-1})[x_1 \cdots x_{k-2}][x_{k+l+2} \cdots x_n] \\ &\qquad + [x_1 \cdots x_{k-2}][x_{k+l+3} \cdots x_n] - [x_1 \cdots x_{k-3}][x_{k+l+2} \cdots x_n]\} \\ &\qquad \times \{(x_{k+l} - x_k)[x_{k+1} \cdots x_{k+l-1}] + [x_{k+1} \cdots x_{k+l-2}] \\ &\qquad - [x_{k+2} \cdots x_{k+l-1}]\}. \end{aligned}$$

Now suppose, e.g., $x_{k+l-1} > x_{k-1}$. Then the first factor is $((x_{k+l-1} - x_{k-1})x_{k-2} - 1)[x_1 \cdots x_{k-3}][x_{k+l+2} \cdots x_n]$ + positive terms, which is positive if all x_i are positive and if $x_i - x_j > 1/x_k$ for $x_i > x_j$ and $k \neq i, j$; but this is even a little weaker than our assumption. Proceeding similarly for $x_{k-1} > x_{k+l-1}$, and for $x_{k+l} \gtrless x_k$ we obtain $\operatorname{sgn}(F_\gamma(x) - F_{\gamma,\sigma}(x)) = \operatorname{sgn}(x_{k+l+1} - x_{k-1})(x_{k+l} - x_k)$.

We may define a cyclic case as follows. The Gauss brackett $[x_1, \cdots, x_n]$ is the sum of all even-gapped sub-products of $x_1, \cdots, x_n$ (that is, those products obtained from $x_1, \cdots, x_n$ by omitting groups of even length of consecutive terms). Let $[x_1, \cdots, x_n]_c$ denote the sum of all even-gapped sub-products where $x_1, \cdots, x_n$ are arranged cyclically rather than linearly. Our lemma remains valid and the maximizing and minimizing arrangements are again those found in §2.

If we denote by $D_{\max}^{(n)}$ the maximal denominator obtained for $\{a_1, \cdots, a_n\}$ then we can obtain the recursion relation

$$D_{\max}^{(n)} = a_n D_{\max}^{(n-1)} + a_{n-1} D_{\max}^{(n-3)} + D_{\max}^{(n-4)}.$$

Thus if the a_n form a numerically periodic sequence of integers (that is a sequence periodic modulo each integer) then the sequence $D_{\max}^{(n)}$ is numerically periodic [2]. This holds also for $D_{\min}^{(n)}$ even though there is no such simple recursion relation.

Our assumptions on $\{a_i\}$ preclude convergence of D for infinite sequences. We may however ask for the limiting values of

$$D_{\max}^{(n)} \Big/ \prod_{i=1}^{n} a_i \text{ and } D_{\min}^{(n)} \Big/ \prod_{i=1}^{n} a_i.$$

We obtain the following.

THEOREM. *Let* $1 \leqq a_1 < a_2 < \cdots < a_n < \cdots$ *be an infinite sequence with* $a_i + 1 \leqq a_{i+1}$ $(i = 1, 2, \cdots)$, *then*

$$D_{\min}^{(n)} \Big/ \prod_{i=1}^{n} a_i = 1 + O\left(\frac{\log n}{n}\right).$$

Also

$$\lim_{n\to\infty} D_{\max}^{(n)} \Big/ \prod_{i=1}^{n} a_i = L$$

exists but is not independent of the sequence $\{a_i\}$. *We have*

$$L \leqq \lim_{n\to\infty} \frac{1}{n!} [n, n-2, \cdots, n-3, n-1] < e^{5/4}.$$

PROOF. For any rearrangement $\{x_1, \cdots, x_n\}$ of $\{a_1, \cdots, a_n\}$ we have

$$[x_1, \cdots, x_n] \Big/ \prod_{i=1}^{n} a_i = 1 + \sum \frac{1}{x_i x_{i+1}} + \sum_{i+1<j} \frac{1}{x_i x_{i+1} x_j x_{j+1}} \tag{6}$$
$$+ \sum_{i+1<j,<k-1} \frac{1}{x_i x_{i+1} x_j x_{j+1} x_k x_{k+1}} + \cdots.$$

If we set $S=\sum 1/x_i x_{i+1}$ then

$$[x_1,\cdots,x_n]\Big/\prod_{i=1}^{n} a_i \leqq 1+S+\frac{1}{2!}S^2+\frac{1}{3!}S^3+\cdots$$
$$+\frac{1}{[n/2]!}S^{[n/2]}<e^S.$$

From our hypothesis we have $a_k \geqq k$ $(k=1,2,\cdots)$ and from (6) we see that $[x_1\cdots x_n]/\prod_{i=1}^{n} a_i$ is not decreased if a_k is replaced by k. Thus

$$1 \leqq D_{\min}^{(n)}\Big/\prod_{i=1}^{n} a_i \leqq \frac{1}{n!}[1,n,3,\cdots,n-1,2] < \exp S_{\min}$$
$$=1+O\left(\frac{\log n}{n}\right),$$

since we saw in the example of §3 that $S_{\min}=O(\log n/n)$.

As it is obvious from (6) that $D_{\max}^{(n)}/\prod_{i=1}^{n} a_i$ increases with n it follows that

$$\lim_{n\to\infty} D_{\max}^{n}\Big/\prod_{i=1}^{n} a_i = L$$

exists with $L \leqq \lim_{n\to\infty}[n,n-2,\cdots,n-3,n-1]/n! < e^{5/4}$, since, according to the example, we have $S_{\max}<5/4$.

The value of $L_1=\lim\ [n,n-2,\cdots,n-3,n-1]/n!$ can be computed and one gets $L_1=2.60\cdots$. We do not know the arithmetic properties of L_1.

5. **Open questions.**

1. For the extremal problems of §2 we may consider graphs of other types such as disconnected graphs or graphs of nonlinear type. As an example we may cite a plane lattice in which each interior point has four neighbors. What are the extremal arrangements when $f(x,y)=xy$, $a_k=1/k$ as in the example of §3?

2. Instead of sums over neighboring pairs as in §3 we may consider sums over consecutive triples, etc. For example, we may ask for the arrangement of $\{1/k\}$ which maximizes the sum of products of 3 consecutive terms. It is reasonable to conjecture that this is the arrangement given in §3.

3. It would be interesting to know for what sets of points in the plane the four point condition (5) can be satisfied by suitable labeling.

References

1. I. Heller, *The travelling salesman's problem* I, George Washington University, Logistics Research Project Report, 1954, 88 pp.
2. E. G. Straus, *Functions periodic modulo each of a sequence of integers*, Duke Math. J. vol. 19 (1952) pp. 379–395.
3. F. Supnick, *On the dense packing of spheres*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 14–26.
4. ———, *Optimal closed paths through certain point sets*, Bull. Amer. Math. Soc. Abstract 62-1-34.

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CONJUGATES IN DIVISION RINGS

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In this note we prove the

THEOREM. *If in a division ring D an element $a \in D$ has only a finite number of conjugates in D then it has only one conjugate, that is, a is in Z, the center of D.*

This theorem, of course, generalizes the famous theorem of Wedderburn which asserts that a finite division ring is a commutative field; however, since Wedderburn's theorem is used in the proof it does not yield a new proof of the result of Wedderburn. We also exhibit two corollaries to the theorem which may be of some independent interest; the second of these extends the result that a polynomial over a field having more roots than its degree in some extension field must be identically zero to a suitable analogue when the roots lie in a division ring.

PROOF OF THE THEOREM. We use the following convention throughout: if K is a division ring then K' will be the group of its nonzero elements under the multiplication of K.

Let $a \in D$ have a finite number of conjugates in D. Thus if $N = \{x \in D \mid xa = ax\}$ then N is a subdivision ring of D; moreover N' is of finite index in D'. Thus N' has a finite number of conjugates in D'. Consequently N has a finite number of conjugates in D, say $N = N_1, N_2, \cdots, N_k$; of course these N_i's are subdivision rings of D. Since the N_i''s are all of finite index in D' and there are a finite number of them, their intersection, T', is also of finite index in D'; in addition T' is normal in D'. Thus T, the intersection of the N_i is a subdivision

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