ON THE LOCATION OF THE ZEROS OF CERTAIN ORTHOGONAL FUNCTIONS[1]

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Let R be a finite region of the z-plane and let $L^2(R)$ denote the class of functions $f(z)$ each analytic in R with $\iint_R |f(z)|^2 dS < \infty$. Let the points $\beta_1, \beta_2, \beta_3, \cdots$ be given interior to R and let $\phi_n(z)$ be that function of class $L^2(R)$ for which $\phi_n(\beta_1) = \phi_n(\beta_2) = \cdots = \phi_n(\beta_{n-1}) = 0$, $\phi_n(\beta_n) = 1$, and which minimizes $\iint_R |\phi_n(z)|^2 dS$ over the class $L^2(R)$. If the points $\beta_1, \beta_2, \beta_3, \cdots$ are not all distinct these requirements of interpolation on $\phi_n(z)$ are to be interpreted in the usual way in the theory of interpolation, to refer to the vanishing of suitable derivatives of $\phi_n(z)$ in multiple points β_k.

The purpose of this note is to establish results on the location of zeros of the functions $\phi_n(z)$. Our main result is the

THEOREM. *Let R be a finite region of the z-plane which contains in its interior the points $\beta_1, \beta_2, \beta_3, \cdots$ and all limit points of the β_n. Then any circle which together with its interior lies in R and which contains in its interior all limit points of the β_n, contains in its interior no zero of $\phi_n(z)$ (other than β_k, $k = 1, \cdots, n-1$) for n sufficiently large.*

The functions $\phi_n(z)$ were first introduced by Bergman [1] in the case that β_n is independent of n, and were later studied by Walsh and Davis [6] in the case that the β_n approach a limit. The totality of zeros of such functions were first studied by the present writers [2] in the analogous case that the double integral over R is replaced by a line integral over the boundary of R; our theorem and other results on the totality of zeros of the $\phi_n(z)$ are of particular significance with reference to the asymptotic properties of the functions, and to the divergence of series $\sum_1^\infty a_n \phi_n(z)$.

The existence of the functions $\phi_n(z)$ follows readily from the theory of normal families, and uniqueness is also easily proved [2]. We introduce the normalized functions $\phi_n^*(z) \equiv \phi_n(z) / [\iint_R |\phi_n(z)|^2 dS]^{1/2}$.

A rapid sketch of the ideas underlying the proof is as follows. The functions $\phi_n^*(z)$ are bounded in norm in R, hence uniformly bounded in absolute value on any closed set interior to R. The fact that $\phi_n^*(z)$ has $n-1$ zeros in R but not near the boundary of R implies that

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$\phi_n^*(z) \to 0$ uniformly on any closed set in R. Thus the subset of R on which $|\phi_n^*(z)|$ is not small lies near the boundary of R, and the situation is similar to that in which the given functions are defined in terms of a line integral over the boundary of R.

As a first step in the formal proof, we establish the

LEMMA. *Let $\phi(z)$ be analytic and of modulus not exceeding the constant L for $|z| \leqq A$ and let $\phi(\beta_k) = 0$, $k = n_0, n_0+1, \cdots, n-1$, with $|\beta_k| \leqq A\rho$, $\rho < 1$. Then in $|z| \leqq |Z| < A$ we have*

$$|\phi(z)| \leqq L\left(\frac{|Z/A| + \rho}{1 + \rho|Z/A|}\right)^{n-1-n_0}.$$

If $A = 1$ we have by the maximum principle

$$|\phi(z)| \Big/ \prod_{k=n_0}^{n-1}\left|\frac{z-\beta_n}{1-\bar{\beta}_n z}\right| \leqq L \qquad \text{for } |z| \leqq 1,$$

and since [3, p. 290]

$$\left|\frac{z-\beta_k}{1-\bar{\beta}_k z}\right| \leqq \frac{|Z| + \rho}{1+\rho|Z|} < 1, \text{for } |z| \leqq |Z| < 1,$$

the result follows for this case. The conclusion of the lemma for arbitrary A follows immediately by making the transformation $z' = z/A$.

We now turn to the proof of the theorem. If S_1 and S_2 are any two point sets, $d(S_1, S_2)$ shall denote the distance from S_1 to S_2. The closure of S_1 is denoted by $\bar{S}_1$. Let T be any circle (i.e. circumference) which together with its interior D lies in R such that the points β_k, $k \geqq n_0$, lie in D. Since $d(T, C) > 0$, where C is the boundary of R, there exists a constant d such that $d(\beta_n, T) \geqq d > 0$ when $n \geqq n_0$. We now choose a circle T_1 contained in R and containing T in its interior; we consider the integral $\iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 (z-\alpha)^{-1} dS$, where D_1 is the interior of T_1 and α is an arbitrary point in D. We can interpret the conjugate of this integral as the force at $z = \alpha$ due to a spread of nonnegative matter over $R - \bar{D}_1$ which repels according to the law of inverse distance. Since the set $R - \bar{D}_1$ lies exterior to T and α is interior to T, this force is equal to the force at α due to the same total mass concentrated at a suitable point β_n' exterior to T [4, pp. 13, 247]:

$$(1) \qquad \iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 \frac{dS}{z-\alpha} = \iint_{R_1-\bar{D}_1} |\phi_n^*(z)|^2 dS \Big/ (\beta_n' - \alpha).$$

We shall now show that the assumption $\phi_n^*(\alpha)=0$ ($\alpha \neq \beta_k$, for $k=1, 2, \cdots, n$) implies that the point β_n' cannot lie exterior to T when n is sufficiently large, and we are thus led to a contradiction. If $\phi_n^*(\alpha)=0$ the function $(z-\beta_n)\phi_n^*(z)/(z-\alpha)$ when suitably defined at $z=\alpha$ is of class $L^2(R)$ and vanishes in the points $\beta_1, \beta_2, \cdots, \beta_n$ and hence (compare [5]) is orthogonal to $\phi_n^*(z)$. That is to say,

$$\iint_R \frac{z-\beta_n}{z-\alpha} \mid \phi_n^*(z) \mid^2 dS = 0,$$

$$(2) \qquad \iint_R \mid \phi_n^*(z) \mid^2 \frac{dS}{z-\alpha} = \iint_R \mid \phi_n^*(z) \mid^2 dS \Big/ (\beta_n - \alpha) = \frac{1}{\beta_n - \alpha}.$$

We next show that as a consequence of (1), (2), and the lemma we have for given η (>0) and for all n sufficiently large

$$(3) \qquad \left| \frac{1}{\beta_n' - \alpha} \right| < \frac{2}{d(T, T_1)},$$

$$(4) \qquad \left| \frac{1}{\beta_n - \alpha} - \frac{1}{\beta_n' - \alpha} \right| < \eta.$$

Let T_2 be a circle concentric with T_1 which together with its interior lies in R and which contains T_1 in its interior. Since $\iint_R |\phi_n^*(z)|^2 dS = 1$ there exists [3, p. 96] a constant L depending on T_2 but not on $\phi_n^*(z)$ such that $|\phi_n^*(z)| \leqq L$ for z on and interior to T_2. We let T_2 play the rôle of the circle $|z| = A$ in the lemma. It then follows from the lemma that when $n > n_0+1$ there exists a positive constant r (<1) independent of n and of z on $\overline{D}_1$ such that

$$(5) \qquad | \phi_n^*(z) | \leqq L r^{n-1-n_0}, \qquad z \text{ on } \overline{D}_1.$$

The finite region R is contained in a circle of radius M and by (5) there exists an n_1 ($>n_0+1$) such that when $n \geqq n_1$ we have $|\phi_n^*(z)|^2 < 1/2\pi M^2$, z in $\overline{D}_1$. Thus when $n \geqq n_1$ we have

$$\left| \iint_{R-\overline{D}_1} | \phi_n^*(z) |^2 dS - 1 \right| = \left| \iint_{R-\overline{D}_1} | \phi_n^*(z) |^2 dS - \iint_R | \phi_n^*(z) |^2 dS \right|$$
$$= \left| \iint_{\overline{D}_1} | \phi_n^*(z) |^2 dS \right| < \frac{1}{2},$$

or

$$\left| \iint_{R-\overline{D}_1} | \phi_n^*(z) |^2 dS \right| > \frac{1}{2} \qquad \text{when } n \geqq n_1.$$

Also

$$\left|\iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 \frac{dS}{z-\alpha}\right| \leqq \frac{1}{d(T, T_1)} \iint_R |\phi_n^*(z)|^2 dS = \frac{1}{d(T, T_1)}$$

for all n and all α interior to T. Hence, from this last inequality and from (1)

$$\left|\frac{1}{\beta_n' - \alpha}\right| = \left|\iint_{R-\bar{D}_1} [|\phi_n^*(z)|^2/(z-\alpha)] dS\right| \Big/ \left|\iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 dS\right.$$

$$< \frac{2}{d(T, T_1)}, \qquad n \geqq n_1,$$

and (3) is established.

By virtue of (5) there exists an n_2 $(>n_0+1)$ such that when $n \geqq n_2$

$$|\phi_n^*(z)|^2 < \frac{\eta d(T, T_1)}{4\pi M^2}, \qquad z \text{ in } \bar{D}_1. \tag{6}$$

Thus, for z on T_1 and hence for z in D_1 we also have $(n \geqq n_2)$ by (6)

$$\frac{|\phi_n^*(z)|^2}{|z-\alpha|} < \frac{\eta}{4\pi M^2}, \tag{7}$$

since the function $\phi_n^*(z)/(z-\alpha)$ is analytic in $\bar{D}_1$ (when suitably defined for $z=\alpha$). Then, by (6) for $n \geqq n_2$

$$\left|1 - \iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 dS\right| = \left|\iint_R |\phi_n^*(z)|^2 dS - \iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 dS\right|$$

$$= \left|\iint_{D_1} |\phi_n(z)|^2 dS\right| < \frac{\eta d(T, T_1)}{4},$$

and by (7) for $n \geqq n_2$

$$\left|\frac{1}{\beta_n - \alpha} - \iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 \frac{dS}{z-\alpha}\right|$$

$$= \left|\iint_R |\phi_n^*(z)|^2 \frac{dS}{z-\alpha} - \iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 \frac{dS}{z-\alpha}\right|$$

$$= \left|\iint_{D_1} |\phi_n^*(z)|^2 \frac{dS}{z-\alpha}\right| < \frac{\eta}{4}.$$

Hence, when $n \geqq \max(n_1, n_2)$ we have

$$\left|\frac{1}{\beta_n - \alpha} - \frac{1}{\beta_n' - \alpha}\right| = \left|\left[\frac{1}{\beta_n - \alpha} - \frac{1}{\beta_n' - \alpha}\iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 dS\right]\right.$$
$$\left. - \frac{1}{\beta_n' - \alpha}\left[1 - \iint_{R-\bar{D}_1} |\phi_n^*(z)|^2 dS\right]\right|$$
$$< \frac{\eta}{4} + \frac{\eta d(T, T')}{4}\cdot\frac{2}{d(T, T')} < \eta,$$

and (4) is established.

We are now in a position to show that β_n' cannot lie exterior to T. If in (4) we set $\eta = 1/d(T, T_1)$, it follows from (3) and (4) that for n sufficiently large (and independent of α in D) we have $1/|\beta_n - \alpha| < 3/d(T, T_1)$. Since $1/2M < 1/|\beta_n - \alpha|$ for all n sufficiently large and for all α in D, we have by setting $\eta = 1/4M$ in (4) that $1/4M < 1/|\beta_n' - \alpha|$ for n sufficiently large and for all α in D. Thus for all n sufficiently large we have

$$(8) \qquad \frac{1}{4M} < \frac{1}{|\beta_n - \alpha|} < \frac{3}{d(T, T_1)}, \qquad \frac{1}{4M} < \frac{1}{|\beta_n' - \alpha|} < \frac{3}{d(T, T_1)}.$$

Since the function $f(Z) \equiv 1/Z$ is uniformly continuous on any closed bounded set B of the Z-plane not containing $Z = 0$, there corresponds to arbitrary ϵ (>0) a δ (>0) such that for all Z_1 and Z_2 on B with $|Z_1 - Z_2| < \delta$ we have $|1/Z_1 - 1/Z_2| < \epsilon$. We now choose B as the set $1/4M \leqq |Z| \leqq 3/d(T, T_1)$ with $\epsilon = d \leqq d(\beta_n, T)$. In (4) we set $\eta = \delta$, whence it follows that there exists an n_3 such that when $n \geqq n_3$ we have $|1/(\beta_n - \alpha) - 1/(\beta_n' - \alpha)| < \delta$ and such that (8) is valid. Then with $Z_1 = 1/(\beta_n - \alpha)$, $Z_2 = 1/(\beta_n' - \alpha)$ when $n \geqq n_3$ we have $|(\beta_n - \alpha) - (\beta_n' - \alpha)| = |\beta_n - \beta_n'| < d \leqq d(\beta_n, T)$ and hence the points β_n' cannot lie exterior to T when $n \geqq n_3$. Since the choice of n_3 is independent of α interior to T, this contradiction completes the proof of the theorem.

The application of this theorem in the study of asymptotic properties of the $\phi_n^*(z)$ and in the study of divergence of series $\sum_1^\infty a_n \phi_n^*(z)$ is wholly analogous to the treatment previously given [2] and is left to the reader.

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PARTIALLY BOUNDED CONTINUED FRACTIONS

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For each complex number sequence a, $f(a)$ denotes the continued fraction

$$\frac{1}{1}+\frac{a_1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\cdots.$$

The statement that $f(a)$ is *partially bounded*[1] means that the sequence a has a bounded infinite subsequence. If $f(a)$ is partially bounded, the series $\sum|b_p|$ diverges, where $b_1=1$, $a_p=1/b_pb_{p+1}$, $p=1, 2, \cdots$, —a necessary condition for convergence of $f(a)$.

Any continued fraction $f(a)$ such that $\sum|b_p|$ diverges is convergent provided its even and odd parts are absolutely convergent,[2] i.e. provided the series $\sum|f_{2p+2}-f_{2p}|$ and $\sum|f_{2p+1}-f_{2p-1}|$ are convergent, where $\{f_p\}_{p=1}^{\infty}$ is the sequence of approximants. The *simple* convergence of the even and odd parts of $f(a)$, together with the divergence of $\sum|b_p|$, is *not* sufficient for the convergence of $f(a)$, (Theorem 3). However, the simple convergence of the even and odd parts of the partially bounded continued fraction $f(a)$ is sufficient for the convergence of $f(a)$. In fact, we have this theorem:

Theorem 1. *Suppose there is a positive integer k such that the sub-*

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[1] $f(a)$ is called *bounded* if the sequence a is bounded—a condition equivalent to the boundedness of a certain infinite matrix. Cf. H. S. Wall, *Analytic theory of continued fractions*, 1948, p. 110. (Referred to later on as AT.)

[2] Trans. Amer. Math. Soc. vol. 67 (1949) pp. 368–380.