DETERMINANTS OF HARMONIC MATRICES

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This paper is concerned with extensions of a theorem by H. S. Wall (Theorem 3 of [3]): if M is a 2×2 harmonic matrix and F corresponds to M then det $M=1$ only in case $F_{11}=-F_{22}$.

As in [3] let H_n denote the class of $n\times n$ harmonic matrices and Φ_n the class of $n\times n$ matrices F of complex-valued functions from the real numbers, continuous and of bounded variation on every interval, such that $F(0)=0$. In [3] Wall has shown that the Stieltjes integral equation,

$$M(s, t) = I + \int_s^t dF(u)\cdot M(u, t), \tag{1}$$

defines a one-to-one correspondence $M\sim F$ between H_n and Φ_n. In studying this correspondence in a more abstract setting, the present author [1] has obtained the continuous product (or "product integral") representation,

$$M(s, t) = {}_s\prod^t \{I + dF\} \quad \text{when} \quad M \sim F. \tag{2}$$

By using this representation, we now have the following extension of Wall's theorem cited above:

THEOREM 1. *If M is in H_n and $M\sim F$ then*

$$\det M(s, t) = \operatorname{Exp}\left(\sum_1^n [F_{pp}(t) - F_{pp}(s)]\right). \tag{3}$$

PROOF. Let $f=\sum_1^n F_{pp}$, and g be the function from the ordered real number pairs $\{s, t\}$ defined by:

$$\det \{I + F(t) - F(s)\} = 1 + f(t) - f(s) + g(s, t).$$

Let J be a number interval, s and t numbers in J, and b a positive number.

If u and v are real numbers then $g(u, v)$ is a sum of products of n factors, of which at least two have the form $F_{ij}(v)-F_{ij}(u)$. Thus there exist a natural number N and a sequence $\{r_p, h_p, k_p\}_1^N$ such that if u and v are numbers in J then

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$$g(u, v) = \sum_1^N r_p(u, v)[h_p(v) - h_p(u)][k_p(v) - k_p(u)],$$

where each r_p is a bounded function from $J \times J$ to the numbers, each h_p is one of the F_{ij}, and each k_p is one of the F_{ij}. From the uniform continuity of the h_p on J and the bounded variation of the k_p on J, it follows that there exists a positive number c such that if $\{u_i\}_0^m$ is a monotone number sequence and $u_0 = s$ and $u_m = t$ and $|u_i - u_{i-1}| < c$ for $i = 1, \cdots, m$ then $\sum_1^m |g(u_{i-1}, u_i)| < b$: hence,

$$\begin{aligned} &\left| \prod_1^m \{1 + f(u_i) - f(u_{i-1})\} - \det \prod_1^m \{I + F(u_i) - F(u_{i-1})\} \right| \\ &\quad \leqq \sum_1^m |g(u_{i-1}, u_i)| \operatorname{Exp}\left(\sum_1^m |f(u_i) - f(u_{i-1})| \right. \\ &\qquad\qquad\qquad\qquad \left. + \sum_1^m |f(u_i) - f(u_{i-1}) + g(u_{i-1}, u_i)| \right) \\ &\quad \leqq b \operatorname{Exp}\left(b + 2 \sum_1^n \int_s^t |dF_{pp}| \right). \end{aligned}$$

Formula (3) is now apparent, since ${}_s\prod^t \{1 + df\} = \operatorname{Exp}(f[t] - f[s])$.

REMARK. The formula (3) is a generalization of the well-known exponential form of the Wronskian of a fundamental set of solutions for an nth order linear differential equation.

It seems natural to ask for a similar result in the case of quasi-harmonic matrices [2]—the statement that the $n \times n$ matrix M is *quasi-harmonic* means that M is an $n \times n$ matrix of complex-valued functions from the ordered pairs $\{s, t\}$ of real numbers, which, for each t, are of bounded variation in s on every interval and which are quasi-continuous in t for each s, and that, for each ordered triple $\{r, s, t\}$ of real numbers, $M(r, s) \cdot M(s, t) = M(r, t)$ and

$$\begin{aligned} M(s, s) &= \frac{1}{2}[M(s-, s) + M(s, s-)] \\ &= \frac{1}{2}[M(s, s+) + M(s+, s)] = I. \end{aligned} \tag{4}$$

Let QH_n denote the class of $n \times n$ quasi-harmonic matrices and $Q\Phi_n$ the class of $n \times n$ matrices F of complex-valued functions from the real numbers, of bounded variation on every interval, such that

$$[F(r) - F(r-)]^2 = [F(r+) - F(r)]^2 = F(0) = 0 \qquad \text{for each } r. \tag{5}$$

In [2] we have shown that (1), with mean integrals replacing the Stieltjes integrals used by Wall, defines a one-to-one correspondence $M \sim F$ between QH_n and $Q\Phi_n$ which extends the correspondence established by Wall in [3] and which is also determined by (2).

THEOREM 2. *If M is in QH_n and $M \sim F$ and G is the "continuous part" of F then*

$$\det M(s, t) = \operatorname{Exp}\left(\sum_1^n [G_{pp}(t) - G_{pp}(s)] \right). \tag{6}$$

PROOF. By the "continuous part" of F we mean (as in proof of Theorem 2.4 of [2]) an element G of Φ_n such that, if $r_1, r_2, \cdots$ is a simple number sequence such that if s is a number at which F is not continuous then there is a natural number k such that $r_k = s$, there is a sequence $F_1, F_2, \cdots$ of elements of $Q\Phi_n$ such that

(i) $F_1 = G$ and, if j is a natural number and $[a, b]$ is a number interval which does not contain r_j, then $F_{j+1}(b) - F_{j+1}(a) = F_j(b) - F_j(a)$ and $F_{j+1}(r_j) - F_{j+1}(r_j -) = F(r_j) - F(r_j -)$ and $F_{j+1}(r_j +) - F_{j+1}(r_j) = F(r_j +) - F(r_j)$, and

(ii) $F_k(s) \to F(s)$ as $k \to \infty$ for each real number s.

Let $F_1, F_2, \cdots$ be such a sequence and $M_k(s, t) = {}_s\prod^t \{I + dF_k\}$ for each natural number k.

If A is an $n \times n$ matrix of complex numbers and $A^2 = 0$ then $\det \{I + A\} = 1$. This may easily be seen as follows: let P be the function from the complex numbers defined by $P(z) = \det \{I + zA\}$; now P is a polynomial satisfying the identity $P(z)P(-z) = 1$, so that P has no zero; hence P is constant and its only value is $P(0)$, which is 1.

Thus we see that $\det M_k(s, t) = \det M_1(s, t)$ for $k = 1, 2, \cdots$. The formula (6) now follows from Theorem 2.5 of [2] and Theorem 1 of the present paper.

BIBLIOGRAPHY

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