A NOTE ON SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

W. R. UTZ[1]

In this note we prove a theorem on the behavior, as $t \to \infty$, of solutions of the nonlinear equation

$$x'' + \alpha(x)x' + \beta(x)x = 0 \tag{1}$$

and prove a corollary to this theorem concerning the behavior of the solutions, as $t \to \infty$, of

$$x'' + g(x') + cx = 0. \tag{2}$$

It will be convenient to be able to refer to the following elementary and intuitively obvious lemma in both proofs.

LEMMA. *Suppose that $x(t)$ is a real function for which $x''(t)$ is defined for all $t \geqq a$.* (i) *If for all $t \geqq a$, $x'(t) < 0$ and $x''(t) \leqq 0$, then* $\lim_{t\to\infty} x(t) = -\infty$. (ii) *If for all $t \geqq a$, $x'(t) > 0$ and $x''(t) \geqq 0$, then* $\lim_{t\to\infty} x(t) = \infty$.

Throughout the note a function is said to oscillate or be oscillatory when and only when it has arbitrarily large zeros.

THEOREM. *If $\alpha(x)$ and $\beta(x)$ are real functions such that for all real x,*

$$\alpha(x) \leqq 0, \qquad \beta(x) > 0$$

and if $x(t)$ is a solution of (1) *valid for all large t, then $x(t)$ oscillates or, for all large t, $x(t)$ is monotone. In case $x(t)$ is monotone increasing,* $\lim_{t\to\infty} x(t) > 0$ *and in case $x(t)$ is monotone decreasing,* $\lim_{t\to\infty} x(t) < 0$.

PROOF. Suppose that $x(t)$ does not oscillate. Then for large t, x is of fixed sign. We assume that $x > 0$ and note that a parallel argument holds for $x < 0$. If $x'(t) = 0$, then $x'' = -\beta(x)x$ and $x'' < 0$ hence $x'(t)$ cannot have arbitrarily large zeros as $x(t)$ would have infinitely many critical values all of which would be maxima. Thus, for t large, $x'(t)$ is of fixed sign.

CASE 1. If $x' < 0$ then, since

$$x'' = -\alpha(x)x' - \beta(x)x,$$

Presented to the Society, August 24, 1956; received by the editors December 27, 1955 and, in revised form, January 23, 1956.

[1] This research was supported by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command.

$x''<0$ and, by the lemma, $\lim_{t\to\infty} x(t) = -\infty$ which contradicts $x(t)>0$ for large t.

CASE 2. If $x'>0$ then $x(t)$ is a montone increasing function and since $x>0$, $\lim_{t\to\infty} x(t)>0$.

COROLLARY. *Let c be a positive constant and suppose that $g(0)=0$ and that for all real z, $g'(z)\leqq 0$. If $x(t)$ is a solution of* (2) *valid for all large t, then $x(t)$ is oscillatory,* $\lim_{t\to\infty} x(t) = \infty$, *or* $\lim_{t\to\infty} x(t) = -\infty$.

PROOF. If we set $v=x'$, then

$$v'' + g'(v)v' + cv = 0 \tag{3}$$

which is (1) with

$$g'(v) = \alpha(v) \leqq 0, \qquad c = \beta(v) > 0$$

hence our theorem applies to (3).

If $v\equiv 0$, then $x(t)\equiv 0$ since $g(0)=0$. Hence x is oscillatory.

Suppose that v is oscillatory but x is not oscillatory. Then v has arbitrarily large zeros and x is eventually of fixed sign. When $v=0$, $x''=-cx$ and x has extrema for arbitrarily large t which are all maxima or all minima. As this is impossible we conclude that v oscillatory implies x oscillatory.

According to our theorem, if v is not oscillatory, then v is monotone. Assume v is monotone increasing, then, according to the theorem above, eventually $v>0$. Thus for sufficiently large t, $x'>0$, $x''\geqq 0$ and by the lemma $\lim_{t\to\infty} x(t) = \infty$.

Similarly, if v is monotone decreasing, $\lim_{t\to\infty} x(t) = -\infty$. This completes the proof of the corollary.

Equations (1) and (2) may be considered as generalizations, in two directions, of the linear equation

$$x'' + dx' + ex = 0 \tag{4}$$

with constant coefficients $d\leqq 0$, $e>0$. Thus we have shown that, for large t, solutions of (1) and (2), under the hypotheses of the theorem and corollary, behave as the solutions of (4).

UNIVERSITY OF MISSOURI AND
INSTITUTE FOR ADVANCED STUDY