ON THE LEFT NUCLEUS OF A BRUCK RING

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Let F be a field of characteristic two and let R be the set of all couples (f, g) with f and g in F. Define equality and addition in R componentwise, and define multiplication by

$$(f, g)(h, k) = (fh + g\cdot k\theta, fk + gh), \tag{1.1}$$

where θ is an additive endomorphism of F. R is a right alternative ring which is alternative if and only if $k\theta = km$, for all k in F and for some fixed m in F. R is a division ring if and only if, for every f in F, the mapping $\mu(f)$, defined by

$$x\mu(f) = x\theta + xf^2, \qquad \text{all } x \text{ in } F, \tag{1.2}$$

is one-to-one of F upon F. (See [2],[1] Appendix.) R. H. Bruck has shown that fields F having additive endomorphisms θ satisfying (1.2), and yet not right multiplications, actually exist. The author, in [1], has therefore called a not alternative, right alternative division ring B (necessarily of characteristic two) a *Bruck ring* if it is two dimensional over some field F and if multiplication in B is given by (1.1). The field F turns out to be the left nucleus of B and, in all the examples given by Bruck, is a simple transcendental extension of a field of characteristic two, and thus imperfect.

In this note we give an elementary proof that this situation must of necessity occur:

THEOREM. *The left nucleus of a Bruck ring is always an imperfect field.*

PROOF. If B is a Bruck ring with left nucleus F, assume that F is perfect. Let θ be the additive endomorphism of (1.1) and suppose that $x\theta = y$, $x \neq 0$ in F. Then $y \neq 0$ and $y = xz$, some z in F. But then $z = v^2$, some v in F, so that $x\theta = xv^2$ and $x\theta + xv^2 = 0$. However, $0\theta + 0v^2 = 0$ and thus the mapping $\mu(v)$, defined by (1.2), is not one-to-one of F upon F. Hence B is not a division ring, a contradiction. It must therefore be true that F is imperfect, which is what we wanted to prove.

We remark that the following corollary answers a question raised by Kleinfeld in a letter to the author dated May, 1953 and mentioned by the author in [1] as an unsolved problem.

COROLLARY. *There exist no finite Bruck rings.*

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[1] Numbers in brackets refer to the bibliography.

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THE EXISTENCE OF OUTER AUTOMORPHISMS OF SOME GROUPS

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F. Haimo and E. Schenkman [1] raised the question: Does a nilpotent group G always possess an outer automorphism? The answer is in the affirmative if G is finite and nilpotent of class 2, as is seen from a Schenkman's [1] stronger result. The object of this note is to show that the answer is also in the affirmative for another family of nilpotent groups, namely the family of all finite p-groups G of order greater than 2 such that $x^p = e$ for every element x in G. Actually, our result is somewhat stronger:

THEOREM 1. *Suppose that G is a group every element of which is of order a divisor of a fixed integer $n>1$. If G has a normal subgroup N such that the factor group G/N is cyclic of order n and such that the intersection $N\cap Z$ of N with the center Z of G contains an element a_0 of order n, then G possesses an outer automorphism which induces identity automorphisms on both N and G/N.*

For the proof we need the following

LEMMA. *If elements $a, a_0, a_1, \cdots, a_k$ in a group G are such that $aa_0 = a_0a$, $a_ia_j = a_ja_i$ for $i, j = 0, 1, \cdots, k$ and such that*

$$a_iaa_i^{-1} = aa_{i-1} \qquad (i = 1, 2, \cdots, k) \tag{1}$$

then we have, for $r = 1, 2, \cdots,$

$$(aa_k)^r = a^r a_k^{C_{r,1}} a_{k-1}^{C_{r,2}} \cdots a_{k-r+1}^{C_{r,r}}, \tag{2}$$

where we set $a_{-1} = a_{-2} = \cdots = e$. If, moreover, a_0 is of order n and if $a_i^n = e$ for $i = 1, 2, \cdots, k$, then

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