

THE FUNCTIONAL DIFFERENTIAL EQUATION $Df = 1/ff$

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Throughout this paper the adjuxtaposition of two functions will denote the substitution of the second into the first. Hence the dot will never be omitted in products of functions. Further, the symbol j will stand for the identity function; that is $j(x) = x$ for every x .¹

With this notation, differentiation of the composite function fg assumes the form $D(fg) = (Df)g \cdot Dg$ where $(Df)g$ is the result of substituting the function g into Df . Let f^{-1} be the inverse function of f . Then $f^{-1}f = j$, and

$$(1) \quad D(f^{-1}f) = 1 = (Df^{-1})f \cdot Df; \text{ hence } Df = 1/(Df^{-1})f.$$

The purpose of this note is to study the existence of analytic solutions to the equation

$$(2) \quad Df = 1/ff.$$

If f^{-1} exists, and is substituted into both sides of (2), then by (1),

$$(3) \quad Df^{-1} = f.$$

Consider analytic solutions of (2)–(3) which leave fixed the (complex) number k . Such a solution will be denoted by f_k ; that is

$$(4) \quad f_k(k) = k.$$

If such a solution exists, then by (2),

$$(5) \quad (Df_k)(k) = 1/k.$$

In view of (4) and (5), let

$$(6) \quad f_k(x) = \sum_{n=0}^{\infty} P_n(k) \cdot (x - k)^n \quad \text{where} \quad P_0(k) = k, \quad P_1(k) = 1/k.$$

To define the remaining $P_n(k)$, consider the expansion of $D^n(hg)$,²

$$(7) \quad D^n(hg) = \sum_{r=1}^n (D^r h)g \cdot \frac{n!}{r!} \sum_{p_1, p_2, \dots, p_r} \frac{D^{p_1} g \cdot \dots \cdot D^{p_r} g}{p_1! \cdot \dots \cdot p_r!}$$

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¹ For the notation throughout, see K. Menger, *Calculus. A modern approach*, Ginn, 1955.

² See for example, note by author, *On the nth derivative of composite functions*, Amer. Math. Monthly vol. 63 no. 5(1956).

where, as throughout this note, $\sum_{n,r}$ is taken over $p_1 + \dots + p_r = n$ and $p_i > 0$. In (7) let $h = f_k^{-1}$ and $g = f_k$. For $n \geq 2$, $D^n(f_k^{-1}f_k) = 0$, while, by (3) $(D^r f_k^{-1})f_k = (D^{r-1}f_k)f_k$. By evaluating both sides of the resulting equation at k one obtains

$$(8) \quad 0 = \sum_{r=1}^n \frac{1}{r} P_{r-1}(k) \cdot \sum_{n,r} P_{p_1}(k) \cdots P_{p_r}(k).$$

In (8), $P_n(k)$ occurs only in the second summation when $r = 1$. Solving for $P_n(k)$ yields the recursion formula:

$$(9) \quad P_n(k) = \frac{-1}{k} \sum_{r=2}^n \frac{1}{r} P_{r-1}(k) \cdot \sum_{n,r} P_{p_1}(k) \cdots P_{p_r}(k).$$

LEMMA 1. *For real $k > 0$, $P_s(k) = (-1)^{s-1} |P_s(k)|$ for $s \geq 1$.*

PROOF. By (6) it is true for $s = 1$. Assume true for $s = 1, 2, \dots, n-1$. To prove for $s = n$, substitute in the recursion formula (9).

Since $(p_1 - 1) + \dots + (p_r - 1) = n - r$, one obtains

$$(10) \quad P_n(k) = \frac{(-1)^{n-1}}{k} \cdot \sum_{r=2}^n \frac{1}{r} |P_{r-1}(k)| \cdot \sum_{n,r} |P_{p_1}(k)| \cdots |P_{p_r}(k)|.$$

Since all terms on the right are positive except for the factor $(-1)^{n-1}$, the lemma is proved.

LEMMA 2. *If (6) converges for real $k > 0$, and $|x - k| < R_k$, then it converges for all real $k' \geq k$, where $R_{k'} \geq R_k$.*

PROOF. Since $P_1(k) = 1/k$, if $k' \geq k$, then $|P_1(k')| \leq |P_1(k)|$.

Assume $|P_s(k')| \leq |P_s(k)|$ for $s = 1, 2, \dots, n-1$. To prove true for $s = n$ in view of (10) is trivial. Hence the series 6 for k is a majorant for the series for any real $k' \geq k$.

Let $\omega_1 = (1 + 5^{1/2})/2$, $\omega_2 = (1 - 5^{1/2})/2$. It may be verified that

$$f_{\omega_1} = \left(\frac{1 + 5^{1/2}}{2} \right)^{(3 - 5^{1/2})/2} \cdot j^{(-1 + 5^{1/2})/2} \text{ and}$$

$$f_{\omega_2} = \left(\frac{1 - 5^{1/2}}{2} \right)^{(3 + 5^{1/2})/2} \cdot j^{(-1 - 5^{1/2})/2}$$

are analytic solutions of (2)–(3) such that $f_{\omega_1}(\omega_1) = \omega_1$ and $f_{\omega_2}(\omega_2) = \omega_2$. The binomial expansion of f_{ω_1} about ω_1 will converge for $|x - \omega_1| < \omega_1$. If the coefficients in this expansion are denoted by A_n , then by the

uniqueness of Taylor Series expansions it follows that $P_n(\omega_1) = A_n$. Hence 6 converges for real $k \geq \omega_1$, and $|x - k| < R_k$ where $R_k \geq R_{\omega_1} = \omega_1$. The following lemma is sufficient to extend convergence to complex k , where $|k| \geq \omega_1$.

LEMMA 3. $(-1)^{n-1} \cdot P_n(k) = k^s \cdot \sum_r a_{n,r} k^r$ where $a_{n,r} > 0$ for $n \geq 1$.

PROOF. Here s , \sum_r^n , and the $a_{n,r}$ are of no concern, merely that $a_{n,r} > 0$. Since $P_1(k) = k^{-1} \cdot 1$, the lemma may be proved by induction using (9).

By Lemma 3, if k is complex then $|P_n(k)| \leq |P_n(|k|)|$. Hence convergence for real $k' > 0$ implies convergence for complex k , $|k| = k'$. Hence (6) converges for all complex k , $|k| \geq \omega_1$.

THEOREM. If (6) converges about $x = k$, then it satisfies equations (2)–(3).

PROOF. If (6) converges, $f_k(k) = k$, and f_k^{-1} exists, analytic about k .

Let $f_k^{-1}(x) = \sum_{n=0}^{\infty} Q_n(k) \cdot (x - k)^n$ where by (6), $Q_0(k) = Q_1(k) = k$. It is sufficient now to show that $Q_s(k) = s^{-1} \cdot P_{s-1}(k)$. For $s = 1$ this follows from comparison of (6) and $Q_1(k)$. To prove for n , assuming true for $s = 1, \dots, n-1$, apply (7) to $f_k^{-1} f_k$ and evaluate at k . Then

$$0 = \sum_{r=1}^n Q_r(k) \cdot \sum_{n,r} P_{p_1}(k) \cdot \dots \cdot P_{p_r}(k).$$

Isolating $Q_n(k)$ yields, since $P_1(k) = 1/k$

$$Q_n(k) = -k^n \sum_{r=1}^{n-1} Q_r(k) \cdot \sum_{n,r} P_{p_1}(k) \cdot \dots \cdot P_{p_r}(k).$$

Substituting $Q_r(k) = P_{r-1}(k)/r$, one obtains in view of (9),

$$\begin{aligned} Q_n(k) &= -k^n \cdot \left\{ -k \cdot P_n(k) - \frac{1}{n} P_{n-1}(k) \cdot \frac{1}{k^n} + k \cdot P_n(k) \right\} \\ &= \frac{1}{n} P_{n-1}(k). \end{aligned}$$

The solutions f_k of (2)–(3) lead to solutions of a class of functional differential equations. Let h be any function having an inverse. Then the $g_{h(k)} = h f_k h^{-1}$ are solutions to the equation

$$(11) \quad Dg = \frac{Dh^{-1}}{(Dh^{-1})g \cdot h^{-1}gg}$$

where $g_{h(k)}(h(k)) = h(k)$. To verify, substitute $g_{h(k)}$ into (11). Then

$$Dg_{h(k)} = (Dh)f_k h^{-1} \cdot (Df_k) h^{-1} \cdot Dh^{-1}.$$

But $(Dh)f_k h^{-1} = (Dh)h^{-1} f_k h^{-1} = 1/(Dh^{-1})g_{h(k)}$. Further $(Df_k)h^{-1} = 1/f_k f_k h^{-1} = 1/h^{-1}g_{h(k)}g_{h(k)}$.

For example, the equation

$$Dg = \frac{a}{gg - b}$$

has as solutions

$$g_{ak+b} = a \cdot f_k \left(\frac{j-b}{a} \right) + b, \quad \text{where } g_{ak+b}(ak+b) = ak+b.$$

In this article it was shown that (6)–(9) converges to a solution of (2)–(3) for all complex k , $|k| \geq \omega_1$. This result has been extended to $|k| > 1$ by considering the solutions of the difference differential equation

$$(12) \quad Dg(z+1) \cdot g(z+2) = Dg(z)$$

in terms of the Dirichlet Series

$$g_b(z) = \sum_{n=0}^{\infty} a_n b^{-nz}, \quad \text{for real } b > 1.$$

If g_b is such a solution of (12) then $f_b = g_b(1+g_b^{-1})$ is a solution of (2)–(3) which is analytic in a neighborhood of b , and for which $f_b(b) = b$. But this approach is altogether different from the above and will be presented in a future article.

Finally, (6)–(9) does not converge for any $x \neq 0$, when $0 < k < 1$. This may be shown by extending Lemma 3, by induction with (10), to the form

$$|P_n(k)| = \sum_{r=0}^{\infty} a_{n,r} k^r / n! k^{(n^2+3n-2)/2} \quad \text{for real } k > 0.$$

Here $\sum a_{n,r} k^r < \sum a_{n,r+1,s} k^r$. Then

$$|P_{n+1}(k)| / |P_n(k)| > 1/(n+1)k^{n+2}$$

which, for $0 < k < 1$, is unbounded.