COMMENT ON "THE DISTANCE TO THE ORIGIN OF A CERTAIN POINT SET IN E^n"

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In a recent paper [1] Karush and Wolfsohn have considered (in slightly different notation) the problem of finding

$$M = \min \sum_{i=0}^{m} a_i^2.$$

Subject to the conditions

$$\sum_{i=0}^{m} a_i = 1, \qquad \sum_{i=0}^{m} i^k a_i = 0 \qquad \text{for } k = 1, 2, \cdots, r \tag{1}$$

where r is a fixed integer between 0 and $m-1$ inclusive. After somewhat lengthy calculations they obtain the result

$$M = 1 - \frac{(m!)^2}{(m+r+1)!(m-r-1)!}. \tag{2}$$

The purpose of this note is to obtain the result by simpler, more geometrical considerations; in particular we employ the theory, treated for example in [2, §72 et seq.], of polynomials orthogonal over a finite set.

Let $P_{k,m}(x) \equiv P_k(x)$ be the orthogonal polynomials of degrees $k=0, 1, \cdots, m$ on the set $x=0, 1, \cdots, m$ having unit constant terms; thus $\sum_{k=0}^{m} P_k(x)P_j(x)=0$ if $k \neq j$, or, in vector notation, $(P_k, P_j)=0$ if $k \neq j$, where P_k denotes the vector with components $P_k(x)$, $x=0, 1, \cdots, m$.

Let Q_k be the corresponding unit vectors, $Q_k = P_k/\|P_k\|$. The squares of the normalizing constants can be written (cf. [2, p. 268, eq. (9)])

$$\frac{1}{\|P_k\|^2} = \frac{2k+1}{m+k+1} \frac{(m!)^2}{(m+k)!(m-k)!}.$$

Since $P_k(0)=1$ we have

$$Q_k(0)^2 = \frac{2k+1}{m+k+1} \frac{(m!)^2}{(m+k)!(m-k)!}.$$

Let $a=(a_0, a_1, \cdots, a_m)$ be a vector satisfying (1). The inner product of a with any polynomial Q of degree $\leqq r$ is clearly given by

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$$(a, Q) = Q(0),$$

and in particular, for the orthonormal polynomials Q_k,

$$(a, Q_k) = Q_k(0), \qquad k = 0, 1, \cdots, r.$$

Now the vectors Q_k, $k=0, 1, \cdots, m$, span E^{m+1} and are orthonormal. Hence any vector $a \in E^{m+1}$ may be written

$$a = \sum_{k=0}^{m} \alpha_k Q_k, \qquad \alpha_k = (\alpha, Q_k)$$

with

$$\|a\|^2 = \sum_{i=0}^{m} a_i^2 = \sum_{k=0}^{m} \alpha_k^2.$$

Then the solution to the minimum problem is obtained by choosing

$$\alpha_k = Q_k(0) \qquad \text{for } k = 0, 1, \cdots, r,$$
$$\alpha_k = 0 \qquad \text{for } k = r+1, \cdots, m.$$

Hence

$$\begin{aligned}
M &= \sum_{k=0}^{r} Q_k(0)^2 \\
&= \sum_{k=0}^{r} \frac{2k+1}{m+k+1} \frac{(m!)^2}{(m+k)!(m-k)!} \\
&= \sum_{k=0}^{r} \frac{(m+k+1)-(m-k)}{m+k+1} \frac{(m!)^2}{(m+k)!(m-k)!} \\
&= \sum_{k=0}^{r} \frac{(m!)^2}{(m+k)!(m-k)!} - \sum_{k=0}^{r} \frac{(m!)^2}{(m+k+1)!(m-k-1)!} \\
&= \left(\sum_{k=0}^{r} - \sum_{k=1}^{r+1}\right)\left(\frac{(m!)^2}{(m+k)!(m-k)!}\right) \\
&= 1 - \frac{(m!)^2}{(m+r+1)!(m-r-1)!},
\end{aligned}$$

in agreement with (2).

References

1. W. Karush and N. Z. Wolfsohn, *The distance to the origin of a certain point set in E^n*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 323–332.

2. W. E. Milne, *Numerical calculus*, Princeton, 1949.

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