ON COLLINEATIONS OF SYMMETRIC DESIGNS

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This note is concerned with systems known as symmetric balanced incomplete block designs, or (v, k, λ) configurations. (See [1] for background and bibliography.) Following the terminology of projective geometry, the elements and the distinguished subsets will be called points and lines respectively. A collineation is a permutation of the points which preserves the class of lines without regard to order of points within lines.

First will be established:

THEOREM 1. *Any collineation α of a (v, k, λ) configuration permutes the points and the lines in such manner that there exists a one-to-one correspondence between cycles of points and cycles of lines with each pair of cycles of the same length. In particular α fixes equally many points and lines.*

PROOF. Number the points $p_1, \cdots, p_v$ and the lines $l_1, \cdots, l_v$. Let A be the incidence matrix for the design defined by $A=(a_{ij})$, $a_{ij}=1$ if $p_i \in l_j$ and 0 otherwise. α corresponds to a unique pair (P, Q) of permutation matrices such that $PA=AQ$. P and Q permute the points and the lines respectively. A is nonsingular. Thus $A^{-1}PA=Q$. P and Q being similar have the same set of characteristic roots with like multiplicities.

For each divisor d of the order of the permutation P, each primitive dth root of unity is a characteristic root of P with multiplicity equal to the number of cycles of P of length divisible by d. Counting characteristic roots, it follows that P and Q have equally many cycles of any length.

Using the above will be proved:

THEOREM 2. *Any group G of collineations of a (v, k, λ) configuration has equally many transitive sets on the points and on the lines. (Note: a fixed point or line is counted as a transitive set.)*

PROOF. Let g be the order of G. G is represented by pairs (P_t, Q_t), $t=1, \cdots, g$, of permutation matrices, P_t and Q_t on points and lines respectively. The total number of points fixed [2] by all the P_t is rg, where r is the number of transitive sets (counting fixed points) of

Received by the editors July 3, 1956.

[1] This work was performed while under Army Ordnance Corps Contract No. DA-33-019-ORD-1911.

$\{P_t\} \sim G$. Likewise the Q_t fix $r'g$ lines, r' the number of transitive sets of $\{Q_t\} \sim G$. By Theorem 1, each corresponding P_t and Q_t fix equally many points and lines. Summing over the elements of G, $rg = r'g$, so that G has equally many transitive sets of points and lines.

Added February 16, 1957. Another fairly immediate consequence is the following:

THEOREM 3. *If a* (v, k, λ) *configuration with incidence matrix A has a collineation defined by* $PA = AQ$, *where P and Q are permutation matrices, then the configuration possesses an incidence matrix A′ such that* $PA' = A'P$.

PROOF. By Theorem 1, the permutations defined by P and Q are similar, and hence conjugate in the symmetric permutation group of degree v. Thus there exists a permutation matrix R such that $Q = R^{-1}PR$. In turn, $PA = A(R^{-1}PR)$, so that $P(AR^{-1}) = (AR^{-1})P$. AR^{-1} is merely the incidence matrix A with columns permuted; the A' asserted to exist is AR^{-1}.

Theorem 2 has been proved by entirely different methods by D. R. Hughes in a forthcoming paper in the Transactions of the American Mathematical Society.

REFERENCES

1. H. J. Ryser, *Geometries and incidence matrices*, Slaught Papers no. 4, Math. Assn. of America, 1955.

2. A. Speiser, *Theorie der Gruppen von endlicher Ordnung*, 3d ed., Dover, Theorem 102, p. 118.

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