ON THE SOLUTION OF $f(f(z))=e^z-1$ AND ITS DOMAIN OF REGULARITY

ROBERT OSSERMAN

The problem of examining the complex solutions $f(z)$ of the equation

$$f(f(z)) = e^z - 1 \tag{1}$$

was suggested to the author by S. Chowla who, together with Kempner, Rivlin, and Thron, proved that $f(z)$ cannot be an entire function. This result is contained as a special case in two recent papers [**1**; **3**] using the theory of entire functions. The purpose of the present note is to show how the use of elementary geometric methods leads quite easily to a slightly stronger theorem which gives some information on a maximum domain of regularity for $f(z)$. This method may also be applied to the equation

$$f(f(z)) = e^z \tag{2}$$

which was treated in detail by Kneser [**2**] who proved the existence of a solution analytic on the whole real axis.

THEOREM. *Let $z=x+iy$ and let Ω denote an infinite strip $|y|<b$ for some constant $b>\pi$. Let $f(z)$ be a function defined in some domain D such that $\Omega\subset D$ and $f(\Omega)\subset D$. If $f(z)$ satisfies* (1) *throughout D then it cannot be analytic in Ω.*

PROOF. Denote by R the image region under $w=f(z)$ of the strip S: $|y|<\pi$. Then if $\zeta=f(w)=f(f(z))=e^z-1$, the image under $f(w)$ of R must be the region T consisting of the ζ-plane slit along the negative real axis from -1 to $-\infty$. Since the composed map $\zeta=f(f(z))$ is a one-one map of S onto T, the maps f of S onto R and of R onto T must also be one-one. Furthermore, since the given correspondences between S and R and between S and T are conformal, so is the correspondence between R and T.

We note next that $f(0)=0$. Namely, if $f(0)=a$, then $f(a)=f(f(0))=0$, so that $a=f(0)=f(f(a))=e^a-1$. But evaluating the derivative of equation (1) at the points 0 and a respectively, we find $f'(a)f'(0)=1$ and $f'(0)f'(a)=e^a=a+1$, whence $a=0$.

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We may thus write the power expansion for $f(z)$ at the origin in the form $f(z)=a_1z+a_2z^2+\cdots$, and inserting this in equation (1) shows that $a_1=1$ and all the a_n are real. In other words, $f'(0)=1$ and $f(z)$ is real on the real axis. But the image of the negative x-axis cannot be the whole negative axis in the w-plane, since the composed map $f(f(z))$ must take the negative x-axis onto the interval $(-1, 0)$. Hence $w=f(z)$ must map the negative x-axis onto an interval $(-c, 0)$ such that f: $(-c, 0)\rightarrow(-1, 0)$. This implies that $c>1$.

We can now show explicitly that the function $f(z)$ must have a singularity at the point $z_0=\log(c-1)+i\pi$ of the region Ω. To see this, denote by R' a copy of the region R placed in the ζ-plane. Then we can find a sequence of points ζ_n in the upper half ζ-plane such that $\zeta_n\in R'\cap T$ for all n, and $\zeta_n\rightarrow -c$. Then the corresponding points z_n of S such that $\zeta_n=f(f(z_n))$ will approach z_0. On the other hand, their images $w_n=f(z_n)$ must approach $-\infty$. To see this we need only note that $w_n=f^{-1}(\zeta_n)$, so that if we consider the points $z_n'=w_n$ of the z-plane and the points $w_n'=\zeta_n$ of the w-plane, then $z_n'=f^{-1}(w_n')$, and $w_n'\rightarrow -c$ implies $z_n'\rightarrow -\infty$.

Bibliography

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Stanford University