

ON A PROBLEM OF MONTGOMERY

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1. Let M be a manifold and let G be a compact Lie group acting on M . For $x \in M$, G_x denotes the isotropic subgroup of G at x , that is, the subgroup consisting of all the elements of G which leave x fixed. Then the following problem has been raised by Montgomery (see Ann. of Math. (2) vol. 50 (1949) p. 260): Do there exist a finite number of subgroups of G such that each G_x is conjugate to one of them? The purpose of this note is to show that, if M is compact and differentiability is assumed, then the answer is in the positive. Whether this result remains true without the assumption of differentiability is still unsettled. But if M is not compact, it is known that even for the analytic case the answer is in the negative. We shall give below an example to this effect, which was communicated to the author by Professor Montgomery.

2. THEOREM. *Let M be a differentiable manifold and let G be a compact Lie group which acts differentiably on M . Then for each point p of M there exists a neighborhood U of p and a finite number of subgroups of G such that the isotropic subgroup of G at any point of U is conjugate to one of these subgroups.*

PROOF. The proof of this theorem relies on a lemma by Montgomery, Samelson and the author (see Ann. of Math. (2) vol. 64 (1956) p. 136), which says: If M and G are as in the theorem and p is a point of M , then there is a closed cell K in M and a closed cell Q in G satisfying the following conditions:

- (i) K contains p and Q contains the identity of G .
- (ii) Whenever $g \in G$ and $x \in K$, $g(x) \in K$ if and only if $g \in G_p$.
- (iii) There is a coordinate system on K such that (a) K is a closed spherical neighborhood of p in M and (b) G_p acts orthogonally.
- (iv) The function $(g, x) \rightarrow g(x)$ defines a homeomorphism of $Q \times K$ onto a neighborhood of p .

Suppose that the theorem is false. Then there exists a point z of M such that every neighborhood of z contains an infinite set F such that, whenever $x, y \in F$, G_x and G_y are not conjugate. The totality of such points z is clearly a closed subset S of M invariant under G . Let S' be the subset of S consisting of all the points z such that $\dim G_z = \inf_{x \in S} \dim G_x$. Then there is a point p of S' such that G_p has the least number of components among those G_z , $z \in S'$. Therefore for any z in S , G_z is not a proper subgroup of G_p .

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According to (iii), there is a coordinate system (x_1, \dots, x_n) on K with p as its origin and such that (a) K is given by $x_1^2 + \dots + x_n^2 \leq 1$ and (b) G_p acts orthogonally. By (b), the fixed point set A of G_p in K is linear and then it may be assumed to be

$$x_1 = \dots = x_r = 0, \quad x_{r+1}^2 + \dots + x_n^2 \leq 1.$$

From (ii), it is easily seen that whenever $x \in K$, $G_x \subset G_p$. Therefore $G_x = G_p$ for every $x \in A$.

Let B be the $(r-1)$ -sphere

$$x_1^2 + \dots + x_r^2 = 1, \quad x_{r+1} = \dots = x_n = 0$$

and let $f: K - A \rightarrow B$ be the map defined by

$$f(x_1, \dots, x_n) = (x_1/d, \dots, x_r/d, 0, \dots, 0),$$

$$d = (x_1^2 + \dots + x_r^2)^{1/2}.$$

For any $x \in K$, both G_x and $G_{f(x)}$ are contained in G_p . It follows from (b) that $G_x = G_{f(x)}$. Since p is a point of S and $Q(K)$ is a neighborhood of p , there exists, by definition, an infinite set F in $G(K)$ such that, whenever $x, y \in F$, G_x and G_y are not conjugate. Let j be the projection of $Q(K)$ on K (see (iv)). It is clear that for any $x \in Q(K)$, $G_{j(x)}$ and G_x are conjugate. Therefore $j(F)$ is an infinite set such that whenever $x, y \in j(F)$, G_x and G_y are not conjugate. Since for every $x \in A$, $G_x = G_p$ and for every $x \in K - A$, $G_x = G_{f(x)}$, it follows that $F' = f(j(F) - A)$ is an infinite set having the same property. F' is a subset of B and then has a limit point z . By definition, z belongs to S . Since $z \in K - A$, G_z is a proper subgroup of G_p . This contradicts our choice of p . Hence the theorem is proved.

COROLLARY. *Let M and G be as in the theorem. If M is compact, then there exist a finite number of subgroups of G such that the isotropic subgroup of G at any point of M is conjugate to one of them.*

3. Now we are in a position to construct a noncompact analytic 3-manifold M on which there is a circle group G of analytic transformations such that every finite subgroup of G is a G_x for some x in M . As mentioned above, this example is due to Professor Montgomery.

Roughly speaking, M is obtained as follows: Cut a sequence of mutually disjoint open anchor rings off the euclidean 3-space, "twist" them and then piece them back to their complement. It is done in such a way that, if it is done for only one of these open anchor rings, the obtained space has a lens space as its one-point-compactification.

We shall use the same symbol to denote a congruence class mod 2π and any real number in this class. The additive group of congruence classes mod 2π will be denoted by G .

Let (x, y, z) be coordinates in the euclidean 3-space. For any positive integer n , A_n denotes the open anchor ring

$$((x^2 + y^2)^{1/2} - n)^2 + z^2 < (1/2)^2$$

and B_n the circle $x^2 + y^2 = n^2$, $z = 0$. It is clear that every point of $A_n - B_n$ can be uniquely written

$$((n + r \cos \beta) \cos \alpha, (n + r \cos \beta) \sin \alpha, r \sin \beta),$$

which we shall abbreviate by $[r, \alpha, \beta]$, where $0 < r < 1/2$ and $\alpha, \beta \in G$. Let $f_n: A_n \rightarrow A_n$ be the function defined by

$$f_n|B_n = \text{identity};$$

$$f_n[r, \alpha, \beta] = [r, \alpha - (n - 1)\beta, -\alpha + n\beta] \text{ whenever } [r, \alpha, \beta] \in A_n - B_n.$$

Then f_n is one-one and onto (but not continuous at points of B_n) and $f_n|(A_n - B_n)$ is a homeomorphism.

The manifold M to be constructed consists of the same points as the euclidean 3-space E . But it is topologized so that the natural map j of $E - \bigcup_{n=1}^{\infty} B_n$ into M , defined by $j(x) = x$, and every map f_n of A_n (as a subspace of E) into M , defined as above, are open homeomorphisms into. $E - \bigcup_{n=1}^{\infty} B_n$, A_n and $A_n - B_n$ are open subsets of E and then inherit analytic structures from the natural one on E . Since $f_n|(A_n - B_n)$ is analytic for every n , we can easily have an analytic structure on M such that j and f_n are all analytic. Hence M is an analytic manifold.

For each $g \in G$, we define a transformation g on M such that $g|(M - \bigcup_{n=1}^{\infty} B_n)$ is a rotation of angle g and each $g|B_n$ is a rotation of angle ng , where rotations are taken about the z -axis. In order to show that g is analytic with respect to the analytic structure given above, we have only to show that $f_n^{-1}gf_n: A_n \rightarrow A_n$ is analytic for every n . Let the symbol $[r, \alpha, \beta]$ stand for

$$((n + r \cos \beta) \cos \alpha, (n + r \cos \beta) \sin \alpha, r \sin \beta),$$

where $0 \leq r < 1/2$ and $\alpha, \beta \in G$. Then

$$(f_n^{-1}gf_n)[r, \alpha, \beta] = [r, \alpha + ng, \beta + g]$$

and hence our assertion follows. This proves that G acts analytically on M .

By construction, it is clear that whenever $x \in B_n$, G_x is the cyclic subgroup of order n . Hence we have obtained an example as desired.