METRIZATIONS OF PROJECTIVE SPACES

HERBERT BUSEMANN

A two-dimensional G-space,[1] in which the geodesic through two distinct points is unique, is either homeomorphic to the plane E^2 and all geodesics are isometric to a straight line, or it is homeomorphic to the projective plane P^2 and all geodesics are isometric to the same circle, see [1, §§10 and 31].

Two problems arise in either case: (1) To determine the systems of curves (in E^2 or P^2) which occur as geodesics. (2) If the geodesics are (or lie on) ordinary straight lines, can the space be imbedded in a higher-dimensional space with the ordinary straight lines as geodesics? The author solved both these problems for E^2, [1, Theorems (11.2) and (14.8)], but left both open for P^2 [1, Appendix (9) and (10)]. Recently Skornyakov [2] solved the first problem for P^2; he modified the author's basic idea through replacing a summation by an integration, and thus eliminated the singularities which the author's procedure would produce in the case of P^2.

The purpose of this note is to show that a device similar to Skornyakov's can be used to solve Problem (2) for P^n. Our method also provides a much simpler solution of Problem (1). Thus we are going to prove simultaneously:

THEOREM I. *Let the projective space P^n, $n \geqq 2$, be metrized as a G-space such that the geodesics are the projective lines. Then P^n can be imbedded in P^{n+1} (and hence in P^m with $m > n$) such that the metric in P^n is preserved and the geodesics in P^{n+1} are the projective lines.*

THEOREM II (OF SKORNYAKOV). *In P^2 let a system Σ' of curves be given such that each curve in Σ' is a closed Jordan curve and two distinct points of P^2 lie on exactly one curve in Σ'. Then P^2 can be metrized as a G-space such that the curves in Σ' become the geodesics.*

For the proof we pass to the sphere S^n or S^2 as universal covering space of P^n or P^2. An r-dimensional spherical sub-space of S^n with the maximal radius will be denoted as a "*great* S^r." We refer to the hypotheses of the two theorems as *Cases* I *and* II respectively. In Case I there is one great S^n in S^{n+1}, which we denote by Q and which

Received by the editors April 21, 1956.

[1] Except for differentiability hypotheses, a G-space is a complete symmetric Finsler space. For the exact definition see §8 in [1]. The results of [1] will be freely used.

is already metrized such that the geodesics are the great circles in Q; we have to extend the metric from Q to S^{n+1}.

In Case II we obtain from Σ' a system Σ of curves on S^2 which are closed Jordan curves and have the property of containing with any point of S^2 also its antipodal point, because a curve in Σ' does not decompose P^2. We select any curve Q in Σ and two antipodal points w, w' not on Q. By a topological mapping of S^2 on itself we can reach that Q and the curves in Σ through w and w' become ordinary great circles. This normalization implies that with the ordinary two-dimensional measure on S^2 all curves in Σ have measure 0, because a curve in Σ not through w, w' is a closed set and intersects each curve through w and w' exactly twice.

A semi-circle K_p is an arc from p to its antipodal point p' on an ordinary great circle in Case I and on a curve in Σ in Case II. For $x \neq p$, p' there is exactly one K_p which passes through x and which we denote by $K_p(x)$.

Q decomposes S^n, $n \geqq 2$, into two open hemispheres H and H'. For an arbitrary point $p \in H$, its antipode $p' \in H'$ and $x \neq p$, p' we put $x_p = K_p(x) \cap Q$. The mapping $x \rightarrow x_p$ is continuous for $x \neq p$, p'. (For the continuity properties of Σ compare the analogous arguments for E^2 in [1, pp. 57, 58]). Also:

(1) *$x \rightarrow x_p$ maps antipodes in S^{n+1} on antipodes in Q and $x_p = x$ for $x \in Q$. In Case* I *a great circle not through p is mapped by $x \rightarrow x_p$ on a great circle in Q.*

In Case I the metric in Q is given. In Case II we introduce on Q the spherical distance or any other distance that makes Q isometric to a circle such that antipodes in the sense of Q coincide with antipodes on S^2. Let 2λ be the common length (see [1, (31.2)]) of the geodesics in Q in Case I or of Q in Case II. We put

$$f_p(x, y) = x_p y_p \qquad \text{for } x \neq p, p', y \neq p, p'. \tag{2}$$

Then

$$f_p(x, y) = f_p(y, x) \geqq 0 \text{ with equality only for } K_p(x) = K_p(y), \tag{3}$$

$$f_p(x, y) + f_p(y, z) \geqq f_p(x, z); \tag{4}$$

here the equality holds only if $K_p(x) = K_p(y)$ or $K_p(y) = K_p(z)$ or if x_p, y_p, z_p lie on a semicircle in Q and y_p between x_p and z_p.

(5) *With $f_p(x, y)$ as distance any semicircle which contains neither p nor p' is isometric to a segment of length* λ, i.e., to the interval $0 \leqq t \leqq \lambda$ with $|t_1 - t_2|$ as distance.

$$f_p(x, y) = x\, y \qquad \text{for } x, y \in Q. \tag{6}$$

The assertions (5) and (6) follow from (1) and (2). Finally we put

$$f_p(p, x) = f_p(x, p) = f_p(p', x) = f_p(x, p') = 0.$$

Then $f_p(x, y) \leqq \lambda$ for all p, x, y. For fixed x, y the function $f_p(x, y)$ is continuous in p for $p \neq x, x', y, y'$ and is lower semi-continuous at the latter points. Therefore, if we use on H a measure proportional to the ordinary spherical measure so normalized that $\int_H dp = 1$, then

$$\rho(x, y) = \int_H f_p(x, y)dp \tag{7}$$

will exist as a Riemann integral. Because of (6)

$$\rho(y, x) = xy \qquad \textit{for } x, y \in Q. \tag{8}$$

(9) *Any semicircle is with $\rho(x, y)$ as distance isometric to a segment of length λ, hence any great circle (or curve in Σ) is isometric to a circle of length 2λ.*

Let G be the great circle containing the given semicircle K. If $x, y \in K$ then

$$\int_H f_p(x, y)dp = \int_{H-G} f_p(x, y)dp.$$

On the other hand, if $p_i \notin G$, $a_i > 0$, $\Sigma a_i = 1$, $i = 1, 2, \cdots, m$, then (5) implies that K is with the distance $\Sigma a_i f_{p_i}(x, y)$ a segment of length λ. Since $\int_{H-G} f_p(x, y)dp$ can be interpreted as Riemann integral, (9) follows.

$$\begin{aligned} &\rho(x, y) = \rho(y, x) \geqq 0 \textit{ with equality only for } x = y \textit{ and} \\ &\rho(x, y) + \rho(y, z) \geqq \rho(x, y) \end{aligned} \tag{10}$$

follow from (3), (9) and (4). Finally:

(11) $\rho(x, y) + \rho(y, z) > \rho(x, z)$ *if x, y, z do not lie on one great circle.*

For Case I the proof is immediate: For any p not in the great S^2 determined by x, y, z the points x_p, y_p, z_p do not lie on one great circle in Q, hence $x_py_p + y_pz_p > x_pz_p$. Since $f_p(x, y) = x_py_p$ is a continuous function of p, when p is not in the great S^2, the assertion follows from (4) and (7).

In both cases, no two of the points x, y, z can be antipodal because then a great circle containing all three points would exist. Hence x_y and z_y are by (1) not antipodal in Q and we can find a semicircle in Q with end points q, q' say, which contains x_y and z_y as interior points. If p lies on $K_y(q)$ close to y then $y_p=q$ or $y_p=q'$ and x_p, z_p lie close to x_y and z_y, consequently there is a semicircle with y_p as one end point which contains both x_p and z_p, so that $x_py_p+y_pz_p>x_pz_p$.

Since it is clear from the continuity properties of $f_\rho(x, y)$ that the distance $\rho(x, y)$ is topologically equivalent to the spherical distance on S^{n+1} or S^2, our theorems follow from (8), (9) and (11) after identifying antipodal points, see [**1**, pp. 128, 129].

The reasons for the brevity of the present proof of Theorem II as compared to Skornyakov's are: using a double instead of single integral and, principally, metrizing S^2 instead of a euclidean plane obtained from P^2 by cutting it along a curve C in P^2 (traversed twice). Showing that the metric satisfies Theorem II after reidentification of diametrically opposite points on C is the main difficulty in [**2**]. This raises the question whether the proofs of theorems [**1**, (14.8)] corresponding to Theorem I for E^n and [**1**, (11.2)] corresponding to Theorem II for E^2 can be similarly simplified and unified. Topological peculiarities, like asymmetry of the asymptote relation, see [**1**, (23.5)], seem to indicate that some complications are unavoidable.

References

1. H. Busemann, *The geometry of geodesics*, New York, 1955.

2. L. A. Skornyakov, *Metrization of the projective plane in connection with a given system of curves*, (Russian), Izvestiya Akad. Nauk SSSR, Ser. Mat. vol. 19 (1955) pp. 471–482.

University of Southern California