NOTE ON CONNECTEDNESS IN TOPOLOGICAL RINGS

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We consider certain properties of topological rings with identity which can be deduced from connectedness.

The following two statements follow immediately from Kaplansky [**2**, Theorems 1 and 2].

1. A connected compact ring is a zero ring.
2. A connected locally compact ring with identity is a finite dimensional algebra over the reals. In this note we drop the assumption of local compactness. Most of the results are consequences of the following simple lemma.

Let M be a topological module over a topological ring A, i.e. M is a topological abelian group, a module over A, and $(a, m)\to am$ is jointly continuous.

LEMMA 1. *If A is connected and M_0 is a submodule of M then AM_0 is in the component of* 0 *in M_0.*

PROOF. Let $m_0\in M_0$, then $a\to am_0$ is a continuous mapping of A onto a connected subset of M_0. Since for $a=0$ in A, $am_0=0$, this subset contains 0.

The following statements follow immediately from Lemma 1:

1. A unital module over a connected ring with identity has only connected submodules.
2. A connected ring with identity has only connected left (right) ideals.
3. The only discrete left (right) ideal in a connected ring with identity is the zero ideal.

We use the definition of covering space and simple connectedness given by Chevalley [**1**], and we prove the following theorem which is in direct analogy to Proposition 5, p. 53 of [**1**].

THEOREM 1. *If a ring A admits a simply connected covering space (S, f) then S can be made into a ring so that f is a ring homomorphism; furthermore if A has an identity so does S.*

PROOF. Assume that (S, f) covers A and that S is simply connected. Then the product $T=S\times S\times S\times S$ is also simply connected. Define the continuous mapping Ω: $T\to A$ by $\Omega(a, b, c, d)=f(a)f(b)+f(c)-f(d)$. Let 0 be an element of S contained in $f^{-1}(0)$, and let β be a fixed element of S not contained in $f^{-1}(0)$. The simple connected-

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ness of T then implies there is a unique "lifting" of Ω (a continuous mapping of Ω': $T\to S$ such that $f\circ\Omega'=\Omega$) such that $\Omega'(\beta, 0, 0, 0)=0$. We define $-a$ in S to be $\Omega'(\beta, 0, 0, a)$ and $a+b$ to be $\Omega'(\beta,0, a, -b)$. These are continuous operations, S becomes an abelian topological group under them, and f is a group homomorphism of S onto the additive group of A (see Proposition 5, p. 53 of [1]).

We now define a (continuous) multiplication in S by $ab=\Omega'(a, b, 0, 0)$. Note that $f(ab)=f(a)f(b)$.

We show first that $a0=0a=0$ for all a in S. The two continuous mappings θ: $a\to 0$ and θ': $a\to a0$ have the property that $f\circ\theta=f\circ\theta'$ and $\theta(\beta)=\theta'(\beta)$. Thus since S is connected and f is a covering map $\theta=\theta'$, $a0=0$. Also the two mappings θ and θ'': $a\to 0a$ obey $f\circ\theta=f\circ\theta''$ and agree on $a=0$, thus $\theta=\theta''$ and $0a=0$. We next show associativity.

The two continuous mappings Σ: $(a, b, c)\to a(bc)$ and Σ'': $(a, b, c)\to(ab)c$ from the connected space $S\times S\times S$ to S have the property that $f\circ\Sigma=f\circ\Sigma''$ and Σ and Σ'' agree on $(0, 0, 0)$. Thus $\Sigma=\Sigma''$ and $a(bc)=(ab)c$. To show left distributivity use the above argument applied to the two mappings $(a, b, c)\to a(b+c)$ and $(a, b, c)\to ab+ac$ and similarly for right distributivity.

If A has an identity e then the element β may be chosen in $f^{-1}(e)$ and is an identity for S. This is because the two mappings $b\to\beta b$ and $b\to b$ have the same composition with f and agree on $b=0$. A similar argument shows $b\beta=b$. This completes the proof of the theorem.

As an immediate corollary we obtain

THEOREM 2. *A ring with identity which admits a simply connected covering space is already simply connected.*

PROOF. By Theorem 1, the ring admits a covering ring with identity and the covering map becomes a ring homomorphism. But the kernel of this is a discrete ideal and, by the property 3 listed above, must be zero. Thus the covering map is a homeomorphism.

REFERENCES

1. C. Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

2. I. Kaplansky, *Locally compact rings* I, Amer. J. Math. vol. 70 (1948) pp. 447–459.

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