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A T -SYSTEM WHICH IS NOT A BERNSTEIN SYSTEM¹

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(I) Let E be a separable real Banach space, let $\{u_i\}_1^\gamma$ be a fundamental² linearly independent sequence (finite or infinite) of points of E , and for each $0 \leq n < \gamma + 1$, let L_n be the linear extension of $\{u_i\}_1^n$, with $L_0 = \{\phi\}$. Then, $\{u_i\}_1^\gamma$ is called a T -system for E if for each x in E and for each $0 \leq n < \gamma + 1$ there is a unique $P_n(x) = \sum_{i=1}^n x_{i,n} u_i$ in L_n such that $\|x - P_n(x)\| = \inf_{Q \in L_n} \|x - Q\|$. The deviation sequence, $\{\delta_i(x)\}_0^\gamma$, of x relative to $\{u_i\}_1^\gamma$ is defined by the conditions: (1) $|\delta_i(x)| = \|x - P_i(x)\|$ for all $0 \leq i < \gamma + 1$ and (2) for $0 \leq i < \gamma + 1$, if $\delta_i(x) \neq 0$, then $\text{sgn } \delta_i(x) = \text{sgn } x_{i',i'}$, where i' is the least integer greater than i for which $x_{i',i'} \neq 0$. If $\{\delta_i(x)\}_0^\gamma \equiv \{\delta_i(y)\}_0^\gamma$ implies that $x = y$, then the T -system $\{u_i\}_1^\gamma$ is called a Bernstein system for E .

The definitions of T -systems and Bernstein systems were essentially introduced by Kadeč in [4], in generalizing some results of Bernstein [2]. The proofs in [2] and [4] show or can be used to show that any two separable Banach spaces are homeomorphic if they possess Bernstein systems of the same cardinality and that for finite dimensional spaces every T -system is a Bernstein system. Now, given a separable Banach space we can always find a fundamental linearly independent sequence of points in it and by strictly convexifying the space [3] such a sequence becomes a T -system. In [4], Kadeč states as an open question: Is every T -system a Bernstein system? Since every separable infinite dimensional Banach space has a T -system under some equivalent norm, an affirmative answer to this would have proved that all such spaces are homeomorphic.

It is the purpose of this paper to exhibit a T -system which is not a Bernstein system. In (II), we describe such a system for the space c_0 under a norm equivalent to the usual sup norm.

We note that Kadeč [4] showed that c_0 does possess a Bernstein

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² Most terms not defined in the text may be found in Banach [1].

system under a norm equivalent to the sup norm and that in a more recent paper [5] he has modified the T -system method of increasing subspaces to a method of decreasing subspaces to prove that all separable infinite dimensional uniformly convex spaces are homeomorphic. The question as to whether or not every separable infinite dimensional Banach space possesses a Bernstein system is apparently still open. It also appears to be undecided whether or not the sequence of powers $\{x^i\}_0^\infty$ which is a T -system for $C[0, 1]$ under the uniform norm is actually a Bernstein system under that norm.

(II) We consider $(c_0, \|\cdot\|)$, the space of sequences of real numbers converging to 0 with the usual norm, $\|\{x_i\}_1^\infty\| = \sup_i |x_i|$. We now give c_0 a norm due to Nikol'skiĭ [6] which is equivalent to $\|\cdot\|$:

$$\|\|\{x_i\}_1^\infty\|\| = \sup_{0 \leq k < \infty} \left[\frac{\sum_{i=1}^k |x_i|}{k+1} + \sup_{n > k} |x_i| \right].$$

We now consider the sequence $\{u_i\}_1^\infty$ of points of c_0 for which u_i is the sequence null in every term except the i th where it is 1. Since $\{u_i\}_1^\infty$ is a basis for $(c_0, \|\cdot\|)$, it is certainly a fundamental linearly independent sequence in $(c_0, \|\|\cdot\|\|)$. Further, $\{u_i\}_1^\infty$ is a T -system for $(c_0, \|\|\cdot\|\|)$, since for every $x = \sum_{i=1}^\infty x_i u_i$ and for every non-negative n , $\|\|\sum_{i=1}^{n+1} x_i u_i\|\| < \|\|\sum_{i=n}^\infty x_i u_i\|\|$ if $x_n \neq 0$. Thus, $P_n(x) = \sum_{i=1}^n x_i u_i$.

The result to be shown here is:

THEOREM. $(c_0, \|\|\cdot\|\|)$ is a separable infinite dimensional Banach space with a T -system which is not a Bernstein system.

PROOF. We let d be a real number ≥ 4 and construct the sequences $a = \{e_i/i!\}_1^\infty$ and $b = \{f_i/i!\}_1^\infty$ according to: $e_1 = d+1; f_1 = d$; for i even, $e_i = (i-1)e_{i-1} + 1, f_i = e_i + 1$; and for i odd, $i \neq 1, f_i = (i-1)f_{i-1} + 1, e_i = f_i + 1$. The following lemmas will complete the proof of the theorem:

LEMMA 1. a and b are in c_0 .

LEMMA 2. For all $1 \leq i < \infty, \|\|\sum_{j=i}^\infty (e_j/j!)u_j\|\| = \|\|\sum_{j=i}^\infty (f_j/j!)u_j\|\|$.

To prove Lemma 1, it suffices to show that $\lim_{n \rightarrow \infty} g_n/n! = 0$, where $g_1 = d+1$ and $g_i = (i-1)g_{i-1} + 2$. It is easily shown by induction that $g_n/n! \leq (d+2)/n + (2/n)(\sum_{j=1}^{n-1} 1/j!)$, and Lemma 1 follows.

For the proof of Lemma 2, we will need:

(A) For all $1 \leq i < \infty, e_i/i! > e_{i+1}/(i+1)! \geq [i/(i+1)]e_i/i!$ and similarly for the f_i ;

(B) For all $1 \leq i < \infty$ and $m \geq i$,

$$(e_i + e_{i+1})/(i+1)! \geq \left(\sum_{k=i}^m e_k/k! \right) / (m+1) + e_{m+1}(m+1)!,$$

and similarly for the f_i ; and

(C) For all $1 \leq i < \infty$, $(e_i + e_{i+1})/(i+1)! = (f_i + f_{i+1})/(i+1)!$. To prove (A), we simply examine the cases of i even and i odd. For (B), it suffices to show that for each i and for each $m \geq i+1$,

$$\left(\sum_{k=1}^{\infty-1} e_k/k! \right) / m + e_m/m! \geq \left(\sum_{k=i}^m e_k/k! \right) / (m+1) + e_{m+1}/(m+1)!$$

or, equivalently, that $[1/m(m+1)] \sum_{k=i}^{m-1} e_k/k! \geq (e_{m+1} - me_m)/(m+1)!$. Now, by the definition of the e_i we always have $e_{m+1} - me_m \leq m+2$, so we need only show that $[1/m(m+1)] \sum_{k=i}^{m-1} e_k/k! \geq (m+2)/(m+1)!$. This, however, certainly holds if $e_{m-1} \geq m+2$ and the latter can be easily shown by induction. The proof for the similar case with the f_i follows in the same way. We note that for $m=2$, $f_{m-1} = f_1 = d \geq 4 = m+2$ which explains our restriction on d . To prove (C), we need only use the equations defining the e_i and f_i .

Now, to complete the proof of Lemma 2, for $q = \sum_{i=1}^{\infty} q_i u_i$ an element of c_0 , let $q^i = \sum_{j=i}^{\infty} q_j u_j$ and let $\|q\|_k = (\sum_{j=1}^k |q_j|)/(k+1) + \sup_{n>k} |q_n|$. Then, by (A), $\|a^i\|_k = e_i/i!$ for $k \leq i-1$ and $\|a^i\|_k = (\sum_{j=1}^k e_j/j!)/(k+1) + e_{k+1}/(k+1)!$ for $k \geq i$ and similarly for b . But, $\|a^i\| = \sup_k \|a^i\|_k$, and so by (B), $\|a^i\| = (e_i + e_{i+1})/(i+1)!$. Similarly, $\|b^i\| = (f_i + f_{i+1})/(i+1)!$ and by (C), $\|a^i\| = \|b^i\|$ to complete the proof of Lemma 2 and the proof of our theorem.

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