A PROPERTY OF POLYNOMIAL EXTENSIONS OF RINGS

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Let A be a commutative ring with unit element, and $A[x] = A[x_1, \cdots, x_n]$ be a ring of polynomials in indeterminates $x_1, \cdots, x_n$ with coefficients in A. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{p}, \mathfrak{q}$ be ideals in A, and $\phi(x)$ be a polynomial in $A[x]$. The following facts are quite well known except the last one (vii).

(i) $\phi(x) \in \mathfrak{a} \cdot A[x] \rightleftarrows$ all the coefficients of $\phi(x)$ are in $\mathfrak{a}$.

(ii) $\mathfrak{a} \cdot A[x] \cap A = \mathfrak{a}$.

(iii) $(\mathfrak{a} \cap \mathfrak{b}) A[x] = \mathfrak{a} \cdot A[x] \cap \mathfrak{b} \cdot A[x]$.

(iv) If $\mathfrak{q}$ is primary with associated prime $\mathfrak{p}$, then $\mathfrak{q} \cdot A[x]$ is primary with $\mathfrak{p} \cdot A[x]$ as its prime [1, Lemma 14, p. 85].

(v) If $\phi(x)$ is a zero-divisor in $A[x]$, then there exists an element $c \neq 0$ in A such that $c\phi(x) = 0$.

Moreover if A is Noetherian, so is $A[x]$, and

(vi) If $\mathfrak{p}$ is prime, then $\mathfrak{p}$ and $\mathfrak{p} \cdot A[x]$ have the same rank [2, Prop. 5, p. 67].

(vii) If $\mathfrak{q}$ is primary, then $\mathfrak{q}$ and $\mathfrak{q} \cdot A[x]$ have the same length.

We first give a proof of the last assertion (vii), which seems to be less familiar. By passing to the residue ring $A[x]/\mathfrak{q} \cdot A[x]$, we may assume $\mathfrak{q} = (0)$. Then, let $\mathfrak{p}$ be the radical of A, and S be the complement of $\mathfrak{p}$ in A. By forming quotient rings of A and $A[x]$ with respect to S, we may assume A is a primary ring with $\mathfrak{p}$ as its unique prime ideal. Let $A \supset \mathfrak{p} = \mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_l = (0)$ be a composition series of ideals of A. Take arbitrary elements a_i $(i = 1, 2, \cdots, l-1)$ such that $a_i \in \mathfrak{q}_i$, $a_i \notin \mathfrak{q}_{i+1}$; then $\mathfrak{q}_i = (\mathfrak{q}_{i+1}, a_i)$. Suppose $a_i \mathfrak{p} \not\subseteq \mathfrak{q}_{i+1}$, then $a_i b \notin \mathfrak{q}_{i+1}$ for some element b in $\mathfrak{p}$. Then $a_i \in \mathfrak{q}_i = (\mathfrak{q}_{i+1}, a_i b)$, hence $a_i \equiv a_i bc (\mathfrak{q}_{i+1})$ for some element c in A; since $1 - bc$ is a unit, we have $a_i \in \mathfrak{q}_{i+1}$, this is a contradiction. Thus we have $\mathfrak{p}\mathfrak{q}_i \subseteq \mathfrak{q}_{i+1}$.

Consider the series: $\mathfrak{p} \cdot A[x] \supset \mathfrak{q}_2 \cdot A[x] \supset \cdots \supset \mathfrak{q}_l \cdot A[x] = (0)$, which we shall prove to be a composition series of the primary ideal (0). For that purpose, let $\mathfrak{q}^*$ be any primary ideal such that $\mathfrak{q}_i \cdot A[x] \supseteq \mathfrak{q}^* \supset \mathfrak{q}_{i+1} \cdot A[x]$ and take an element $f(x)$ such that $f(x) \in \mathfrak{q}^*$, $f(x) \notin \mathfrak{q}_{i+1} \cdot A[x]$. Then we have $f(x) = g(x) + a_i \phi(x)$, with $g(x) \in \mathfrak{q}_{i+1} A[x]$; hence $a_i \phi(x) \in \mathfrak{q}^*$ and $a_i \phi(x) \notin \mathfrak{q}_{i+1} \cdot A[x]$. Now we assert $\phi(x) \notin \mathfrak{p} \cdot A[x]$; suppose the contrary: $\phi(x) \in \mathfrak{p} \cdot A[x]$, then $a_i \phi(x) \in \mathfrak{q}_i \cdot \mathfrak{p} \cdot A[x] \subseteq \mathfrak{q}_{i+1} \cdot A[x]$, a contradiction. Thus, from $a_i \phi(x) \in \mathfrak{q}^*$ and $\phi(x) \notin \mathfrak{z} \cdot A[x]$ follows $a_i \in \mathfrak{q}^*$, whence $\mathfrak{q}^* = \mathfrak{q}_i \cdot A[x]$. This completes the proof.

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From (vi) and (vii) and elementary properties of quotient rings follows immediately the fact:

Let $A=K[x_1, \cdots, x_n]$ be a polynomial ring over a field K, and let $\mathfrak{p}$ be a prime ideal, $\mathfrak{q}$ be a primary ideal of A. Let $L=K(u)=K(u_1, \cdots, u_m)$ be a purely transcendental extension of K, and form the polynomial ring over L: $R^*=L[x_1, \cdots, x_n]$. Then $\mathfrak{p}\cdot R^*$, $\mathfrak{q}\cdot R^*$ are prime and primary respectively. Moreover we have dim $\mathfrak{p}$ $=$dim $\mathfrak{p}\cdot R^*$ and length $\mathfrak{q}=$length $\mathfrak{q}R^*$.

This is a special case of a well known theorem. To see this, we have only to consider the intermediary ring: $R_1 = K[u][x_1, \cdots, x_n]=R[u]$. This is a polynomial extension of R, and R^* is a quotient ring of R_1.

Finally we shall note another application of (vii). Let Q be a local ring, $\mathfrak{m}$ be its maximal ideal and put $Q/\mathfrak{m}=k$. Form a polynomial extension of Q: $Q[x]$; and form a quotient ring of $Q[x]$ with respect to its prime ideal $\mathfrak{m}\cdot Q[x]$, which we shall denote by Q^*. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of Q, and $\mathfrak{m}^*$, $\mathfrak{q}^*$ be the extensions of $\mathfrak{m}$, $\mathfrak{q}$ in Q^* respectively. Then from (vii) follows that $\mathfrak{q}$ and $\mathfrak{q}^*$ have the same characteristic function, i.e. $P_{\mathfrak{q}}(s)=P_{\mathfrak{q}^*}(s)$ in Samuel's notation [4]. A fortiori, $\mathfrak{q}$ and $\mathfrak{q}^*$ have the same multiplicity: $e(\mathfrak{q})=e(\mathfrak{q}^*)$. We notice the residue field of Q^* is $k(x)$. As is well known, a local ring with a finite residue field is sometimes harder to be dealt with. But the above observation suggests that as long as we concern the multiplicities of $\mathfrak{m}$-primary ideals, there may be some cases where we can overcome the difficulty by passing from Q to Q^*. We shall illustrate this by quoting a concrete example. Let l be the dimension of Q, and assume $l>0$. Let $\mathfrak{n}_1, \cdots, \mathfrak{n}_r$ be all the l-dimensional primary components of the zero ideal of Q. Under the hypothesis that Q and its residue field have the same characteristic, D. G. Northcott and D. Rees [3] proved the following interesting fact:

$$e(\mathfrak{q}) = \sum_{i=1}^{r} e(\mathfrak{q} + \mathfrak{n}_i/\mathfrak{n}_i).$$

Notice that Q^* has the same dimension as Q, the l-dim. primary components of the zero ideal of Q^* are $\mathfrak{n}_1^*, \cdots, \mathfrak{n}_r^*$, where $\mathfrak{n}_i^*=\mathfrak{n}_i\cdot Q^*$, and $e(\mathfrak{q}+\mathfrak{n}_i/\mathfrak{n}_i)=e(\mathfrak{q}^*+\mathfrak{n}_i^*/\mathfrak{n}_i^*)$. Thus, in proving the above fact, we may assume that Q has an infinite residue field; this remark may bring about a considerable simplification to the proof.

References

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CONCERNING REAL VALUED MAPS OF THE n-SPHERE[1]

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A definition for the width of a closed curve, believed to be new, is used together with some standard homology theory, to prove the following theorem: *Let f: $S^n \to E^1$ be continuous, $0 \leqq d \leqq 2$, p the covering map $S^n \to P^n$. If $X_d = \{x \in S^n \mid$ there exists $y \in S^n$ with $\rho(x, y) = d$, $f(x) = f(y)\}$, then pX_d carries the nontrivial* mod 2 *Čech $n-1$ cycle of P^n.* This generalizes Theorem 2 of [4], already considerably extended in other directions by Bourgin [2] and Yang [5].

We will use the following notation: E^{n+1} = Euclidean $n+1$ space, ρ is the Euclidean metric, ω is the origin in E^{n+1}. $S^n = \{x \in E^{n+1} \mid \rho(x, \omega) = 1\}$. P^n is projective n space. p_i: $A_1 \times A_2 \to A_i$, $i = 1, 2$ will denote the projection. $\mathcal{H}_p(A) = p$ dimensional Čech homology group of A with coefficients the integers mod 2 (the only coefficients to be used here). $T^2 = S^1 \times S^1$, Δ = diagonal in T^2. In a space X, $\tilde{A} = X - A$, $\overline{A}$ = closure of A.

LEMMA 1. *Let g: $S^1 \to E^1$ be continuous, X a connected subset of T^2 such that either $p_1X = S^1$ or $p_2X = S^1$. Then there exists $(x, y) \in X$ such that $g(x) = g(y)$.*

PROOF. If $p_1X = S^1$, then there exists $(x_1, y_1) \in X$ such that $g(x_1) = \max_{x \in S^1} g(x)$, and $(x_2, y_2) \in X$ such that $g(x_2) = \min_{x \in S^1} g(x)$. Then if neither $(g(x_1), g(y_1))$ nor $(g(x_2), g(y_2))$ is on the diagonal in E^2, then they are on opposite sides. If $p_1X \neq S^1$, then we may make the above argument on the second coordinate.

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