A CLASS OF INEQUALITIES[1]

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1. **Introduction.** We present inequalities which bound $\max_x |y(x)|$ in terms of integrals involving y and dy/dx. These results appear to be unknown, despite the fact that they are fundamental, useful and are obtained by extremely elementary methods. An inequality similar to (17) below was pointed out to the author by Professor P. C. Rosenbloom; his proof suggested the ideas used here. The author is indebted to Professor Rosenbloom.

2. **Notation and assumptions.** Let a and b be numbers $(a<b)$ which are fixed throughout the rest of this paper. Any function denoted y or $y(x)$ is assumed to be defined and continuous on $a \leqq x \leqq b$ and to have there a derivative which is sectionally continuous. The prime ($'$) will always denote differentiation with respect to x. Let $f(x), f'(x), g(x)$ be given functions which are continuous on $a \leqq x \leqq b$.

3. **General results.** For a fixed t satisfying $a<t<b$, suppose that $w(x; t)$ satisfies the following four conditions:

(1) $(fw')' - gw = 0$ for $a \leqq x < t$ and $t < x \leqq b$,

(2) $w(x; t)$ is continuous for $a \leqq x \leqq b$,

(3) $\lim_{\epsilon \to 0} [w'(t-\epsilon; t) - w'(t+\epsilon; t)]f(t) = 1$,

(4) $f(a)w'(a)y(a) = f(b)w'(b)y(b)$.

Now

$$\int_a^b fw'y'dx = \lim_{\epsilon \to 0} \left[\int_a^{t-\epsilon} fw'y'dx + \int_{t+\epsilon}^b fw'y'dx \right]$$

$$= \lim_{\epsilon \to 0} \left(fw'y \Big]_a^{t-\epsilon} + fw'y \Big]_{t+\epsilon}^b - \int_a^{t-\epsilon} y(fw')'dx \right.$$

$$\left. - \int_{t+\epsilon}^b y(fw')'dx \right) = fw'y \Big]_a^b$$

$$+ y(t) - \int_a^b gwydx.$$

Hence

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$$y(t) = \int_a^b (fw'y' + gwy)dx. \tag{5}$$

Let numbers α, β; γ, δ; p, q; μ, ν satisfy $\alpha+\beta=1$, $\gamma+\delta=1$, $p\geqq 1$, $1/p+1/q=1$; $\mu\geqq 1$, $1/\mu+1/\nu=1$. Then $|fw'y'+gwy| \leqq |f^\alpha w'|\,|f^\beta y'| + |g^\gamma w|\,|g^\delta y| \leqq (|f^\alpha w'|^p+|g^\gamma w|^p)^{1/p}(|f^\beta y'|^q+|g^\delta y|^q)^{1/q}$, so that

$$\begin{aligned} |y(t)| \leqq &\left[\int_a^b (|f^\alpha w'|^p + |g^\gamma w|^p)^{\mu/p} dx\right]^{1/\mu} \\ &\cdot\left[\int_a^b (|f^\beta y'|^q + |g^\delta y|^q)^{\nu/q} dx\right]^{1/\nu}. \end{aligned} \tag{6}$$

For fixed values of α, γ, p, μ this takes the form

$$|y(t)| \leqq M(t)\left[\int_a^b (|f^\beta y'|^q + |g^\delta y|^q)^{\nu/q} dx\right]^{1/\nu}, \tag{7}$$

where

$$M(t) = \left[\int_a^b (|f^\alpha w'|^p + |g^\gamma w|^p)^{\mu/p} dx\right]^{1/\mu}, \tag{8}$$

and does not depend on the function y. Or, letting $M=\sup_{0<t<a} M(t)$,

$$|y(t)| \leqq M\left[\int_a^b (|f^\beta y'|^q + |g^\delta y|^q)^{\nu/q} dx\right]^{1/\nu}. \tag{9}$$

A case of equation (6) of particular interest is when $f(x)\geqq 0$, $g(x)\geqq 0$ and $\mu=p=2=1/\alpha=1/\gamma$. Then (6) takes the form

$$|y(t)|^2 \leqq \tilde{M}^2(t)\int_a^b [(y')^2 f + y^2 g]dx, \tag{6a}$$

where, by the argument leading to equation (5) with y there replaced by w,

$$\begin{aligned} \tilde{M}^2(t) &= \int_a^b [(w')^2 f + w^2 g]dx \\ &= w(t;t) + f(b)w'(b)w(b) - f(a)w'(a)w(a). \end{aligned} \tag{8a}$$

4. **Particular cases.** One can get $M(t)$ in (7) into more concrete form by choosing $f(x)$ and $g(x)$ so that $w(x;t)$ can be found explicitly. If $f(x)\geqq 0$ and $g(x)\geqq 0$, then the computations can be made easier by means of (8a) and one then obtains a concrete form of the inequality (6a). It is a fact, but one which does not seem to be helpful in this

connection, that $\tilde{M}^2(t)=\int_a^b[(w')^2f+w^2g]dx\leqq\int_a^b[(u')^2f+u^2g]dx$ for any function $u(x)$ that agrees with $w(x;t)$ when $x=a$, b and t. This follows from the fact that (1) is the Euler-Lagrange equation for that variational problem.

To guarantee that equation (4) is satisfied we must make certain assumptions about w; these assumptions vary according to the conditions we impose on y and f. Accordingly we treat several cases:

CASE I. Here we assume that $f(x)>0$ for all $a\leqq x\leqq b$. Now we may be interested in inequality (7) for functions y which are required to satisfy: (α) no additional restrictions; (β) $y(a)=ky(b)$ where k is a fixed constant; (γ) $y(a)=y(b)=0$; naturally we expect successively smaller values of $M(t)$ in cases (α), (β), (γ). Since equation (1) is now regular there are functions $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$ which satisfy equation (1) for $a\leqq x\leqq b$ and such that $w_1(a)=w_3(b)=w_2'(a)=w_4'(b)=1$, $w_1'(a)=w_3'(b)=w_2(a)=w_4(b)=0$. The solution $w(x;t)$ is then

$$w(x;t)=\begin{cases}A(t)w_1(x)+B(t)w_2(x), & \text{for } a\leqq x<t,\\ C(t)w_3(x)+D(t)w_4(x), & \text{for } t<x\leqq b,\end{cases}\tag{10}$$

where to satisfy conditions (2) and (3) we must have

$$Aw_1(t)+Bw_2(t)-Cw_3(t)-Dw_4(t)=0,\tag{11}$$

$$Aw_1'(t)+Bw_2'(t)-Cw_3'(t)-Dw_4'(t)=1/f(t).\tag{12}$$

These equations are consistent since the Wronskian w_{21} is not zero, $(w_{ij}=w_i'w_j-w_j'w_i)$. By (8a) we now have

$$\tilde{M}^2(t)=A(t)w_1(t)+B(t)w_2(t)+f(b)C(t)D(t)-f(a)A(t)B(t).\tag{13}$$

CASE Iα. To satisfy equation (4) in case (α) we must require that $w'(a)=w'(b)=0$, which gives the additional equations

$$B=0,\tag{14}$$

$$D=0\tag{15}$$

The four equations (11), (12), (14), (15) serve to determine A, B, C, D. We then get

$$\begin{aligned}w(x,t)&=\frac{w_3(t)w_1(x)}{w_{13}(t)f(t)} && a\leqq x\leqq t,\\ &=\frac{w_1(t)w_3(x)}{w_{13}(t)f(t)} && t\leqq x\leqq b,\end{aligned}$$

provided that $w_{13}(t)\neq 0$.

One can, with sufficient patience, compute solutions $w(x;t)$ for particular functions $f(x)$ and $g(x)$, put the results in (8) and get interesting inequalities. One can, with somewhat less patience, use (8a) and (6a) which become now $\tilde{M}^2(t)=w(t;t)$ and

$$|y(t)|^2 \leqq \frac{w_1(t)w_3(t)}{w_{13}(t)f(t)}\int_a^b [(y')^2f+y^2g]dx, \text{ respectively.}$$

To illustrate: suppose $f(x)=1$, $g(x)=c^2>0$. Then $w_1=\cosh c(x-a)$, $w_3=\cosh c(b-x)$, $w_{13}=c\sinh c(b-a)$. Then

$$w(t;t) = \cosh c(b-t)\cosh c(t-a)/c\sinh c(b-a).$$

One has thus the inequality

$$|y(t)|^2 \leqq \frac{\cosh c(b-t)\cosh c(t-a)}{c\sinh c(b-a)}\int_a^b [(y')^2+c^2y^2]dx \tag{16}$$

valid for any function y; one also has the uniform bound

$$|y(t)|^2 \leqq \frac{\coth c(b-a)}{c}\int_a^b [(y')^2+c^2y^2]dx. \tag{17}$$

Finally we note that here (case Iα) the $\tilde{M}^2(t)$ given by $w(t,t)$ gives the best possible bound in (6a) since equality holds at the point t if $y_t(x)$ is selected to be $w(x;t)$. Thus, for example, equality holds for $y_t(t)$ in (16) if $y_t(x)=\cosh c(b-t)\cosh c(x-a)$, for $a\leqq x\leqq t$, and $\cosh c(b-x)\cosh c(t-a)$ for $t\leqq x\leqq b$. Similarly equality holds for $y_a(a)$ in (17) if $y_a(x)=\cosh c(b-x)$.

CASE Iβ. Now the $w(x,t)$ is given by (10) where $A(t)$, $B(t)$, $C(t)$, $D(t)$ must satisfy equations (11), (12) and, to satisfy (4) when $y(a)=ky(b)$, $f(a)Bk=f(b)D$. This gives

$$w_{13}A = \frac{w_3}{f(t)} + B\left(w_{32}+\frac{kf(a)}{f(b)}w_{43}\right),$$
$$w_{13}C = \frac{w_1}{f(t)} + B\left(w_{12}+\frac{kf(a)}{f(b)}w_{41}\right);$$

inserting these in (13) we get:

$$\begin{aligned} w_{13}\tilde{M}^2(t) &= \frac{w_1w_3}{f(t)} + B\left(w_3w_{12}+\frac{kf(a)}{f(b)}w_1w_{43}+\frac{f(a)}{f(t)}(kw_1-w_3)\right) \\ &\quad + B^2f(a)\left[k\left(w_{12}+k\frac{f(a)}{f(b)}w_{41}\right)-w_{32}-\frac{kf(a)}{f(b)}w_{43}\right] \\ &= \xi+\eta B+\zeta B^2, \end{aligned}$$

which has its extreme value when

$$B = -\eta/2\zeta \text{ and then has the value } \xi - \eta^2/4\zeta \text{ or}$$

$$(18)\quad w_{13}\tilde{M}^2(t) = \frac{w_1 w_3}{f(t)} - \frac{\left[w_3 w_{12} + \dfrac{kf(a)}{f(b)} w_1 w_{43} + \dfrac{f(a)}{f(t)}(kw_1 - w_3)\right]^2}{4f(a)\left[kw_{12} + (k^2)\dfrac{f(a)}{f(b)} w_{41} - w_{32} - \dfrac{kf(a)}{f(b)} w_{43}\right]}.$$

If we consider the same special case as before ($f(x)=1$, $g(x)=c^2>0$), then equation (18) becomes

$$(19)\quad \begin{aligned} &\tilde{M}^2(t) \\ &= \{k^2 \cosh c(t-a)[\cosh c(b-a)\cosh c(b-t) - \cosh c(t-a)] \\ &\quad + \cosh c(b-t)[\cosh c(b-a)\cosh c(t-a) - \cosh c(b-t)]\} \\ &\quad \cdot \{c \sinh c(b-a)[(1+k^2)\cosh c(b-a) - 2k]\}^{-1}. \end{aligned}$$

That is, for all y satisfying $y(a)=ky(b)$, one has

$$|y(t)|^2 \leqq \tilde{M}^2(t)\int_a^b [(y')^2 + c^2y^2]dx,$$

where $\tilde{M}^2(t)$ is given by (19). In particular setting $k=0$ one has that for all y such that $y(a)=0$

$$(20)\quad \begin{aligned} |y(t)|^2 \leqq{}& \frac{\cosh c(b-t)[\cosh c(b-a)\cosh c(t-a) - \cosh c(b-t)]}{c \sinh c(b-a)\cosh c(b-a)} \\ &\cdot \int_a^b [(y')^2 + c^2y^2]dx, \end{aligned}$$

and the uniform bound

$$(21)\quad |y(t)|^2 \leqq \frac{\tanh c(b-a)}{c}\int_a^b [(y')^2 + c^2y^2]dx;$$

or, integrating (20)

$$(22)\quad \int_a^b y^2 dx \leqq \frac{(b-a)\tanh c(b-a)}{2c}\int_a^b [(y')^2 + c^2y^2]dx.$$

The inequalities here are not necessarily best possible, since in the proof of this for case Iα we permitted ourselves to take $y=w$, but now we are requiring that $y(a)=ky(b)$ and w might not satisfy this. In particular letting $c\rightarrow 0$ in (22) we obtain

$$\int_a^b y^2dx \leqq \frac{(b-a)^2}{2}\int_a^b (y')^2dx,$$

and it is known [1, pp. 183–187], that the best constant here is not $(b-a)^2/2$ but $4(b-a)^2/\pi^2$.

CASE Iγ. Now only equations (11) and (12) must be satisfied by A, B, C, D. This leaves a two parameter family of solutions $w(x; t)$ for us to choose among. This enables us to lower the value of $M(t)$. Now we minimize $M^2(t)$ as given by (13), where the A, B, C, D are constrained to satisfy equations (11) and (12). If, for example we take again $f(x)=1$, $g(x)=c^2>0$ then we find the inequality

$$(23)\qquad |y(t)|^2 \leqq \frac{\sinh c(t-a)\sinh c(b-t)}{c\sinh c(b-a)}\int_a^b [(y')^2+c^2y^2]dx$$

valid for any function y such that $y(a)=y(b)=0$; from this the uniform bound

$$(24)\qquad |y(t)|^2 \leqq \frac{\tanh\dfrac{c(b-a)}{2}}{2c}\int_a^b [(y')^2+c^2y^2]dx;$$

or, by integrating (23),

$$(25)\qquad \int_a^b y^2dx \leqq \frac{c(b-a)-\tanh c(b-a)}{2c^2\tanh c(b-a)}\int_a^b [(y')^2+c^2y^2]dx.$$

Again, as in case Iβ we cannot assert that the inequalities are best possible for reasons similar to those given in case Iβ; in particular let $c\to 0$ in (25) to obtain

$$\int_a^b y^2dx \leqq \frac{(b-a)^2}{6}\int_a^b (y')^2dx;$$

it is known that best constant is for the 6 to be replaced by π^2 (Wirtinger's Inequality [1]), so that equality is not achievable here. If we let $c\to 0$ in (23) we obtain

$$|y(t)|^2 \leqq \frac{(t-a)(b-t)}{b-a}\int_a^b (y')^2dx \leqq \frac{b-a}{4}\int_a^b (y')^2dx,$$

for all functions $y(x)$ satisfying $y(a)=y(b)=0$. For the function $y(x)=x-a$ for $a\leqq x\leqq (a+b)/2$; $y(x)=b-x$, $(a+b)/2\leqq x\leqq b$ one has $|y((a+b)/2)|^2=((b-a)/4)\int_a^b(y')^2dx$.

Since $y(a)=y(b)=0$ and for such functions

$$\int_a^b (y')^2 dx$$

$$= -\int_a^b yy''dx = \frac{1}{4}\int_a^b \left[\left(\frac{y}{\epsilon} - \epsilon y''\right)^2 - \left(\frac{y}{\epsilon} + \epsilon y''\right)^2\right] dx$$

$$\leqq \frac{1}{4}\int_a^b \left[\left(\frac{y}{\epsilon} - \epsilon y''\right)^2 + \left(\frac{y}{\epsilon} + \epsilon y''\right)^2\right] dx = \frac{1}{2\epsilon^2}\int_a^b [y^2 + \epsilon^4 (y'')^2] dx$$

it follows from (23)–(25) that

$$|y(t)|^2 \leqq \frac{\sinh c(t-a)\sinh c(b-t)}{2c \sinh c(b-a)} \int_a^b \left[\left(\frac{1}{\epsilon^2} + 2c^2\right) y^2 + \epsilon^2 (y'')^2\right] dx,$$

$$|y(t)|^2 \leqq \frac{\tanh \dfrac{c(b-a)}{2}}{4c} \int_a^b \left[\left(\frac{1}{\epsilon^2} + 2c^2\right) y^2 + \epsilon^2 (y'')^2\right] dx,$$

$$\int_a^b y^2 dx \leqq \frac{c(b-a) - \tanh c(b-a)}{4c^2 \tanh c(b-a)} \int_a^b \left[\left(\frac{1}{\epsilon^2} + 2c^2\right) y^2 + \epsilon^2 (y'')^2\right] dx,$$

for any $\epsilon > 0$ and $y(x)$ such that $y(a) = y(b) = 0$.

CASE II. If we do not have $f(x)$ different from zero throughout the interval then we cannot assert the existence of the functions w_1, w_2, w_3, w_4 used in Case I. However if we are putting no restrictions on y and if e.g., $f(a) = 0$, then condition (4) can be satisfied by taking $w'(b) = 0$ and this gives us another degree of freedom with w; to illustrate: let $f(x) = x^m$, $(m > 0)$, $g(x) = mx^{m-2}$, $a = 0$, $b = 1$. Then an easy computation shows that

$$w(x;t) = \begin{cases} \dfrac{x(mt + t^{-m})}{1+m} & \text{for } 0 \leqq x \leqq t, \\[2ex] \dfrac{t(mx + x^{-m})}{(1+m)} & \text{for } t \leqq x \leqq 1 \end{cases} \tag{26}$$

satisfies (1)–(4). Now, by (8a), $\tilde{M}^2(t) = w(t;t) = t(mt + t^{-m})/(1+m)$, so that we can assert for all functions y

$$|y(t)|^2 \leqq \frac{t(mt + t^{-m})}{1+m} \int_0^1 [x^m (y')^2 + mx^{m-2} y^2] dx; \tag{27}$$

for $0 < m \leqq 1$ we have the uniform bound

$$|y(t)|^2 \leqq \int_0^1 [x^m (y')^2 + mx^{m-2} y^2] dx \tag{28}$$

or, integrating (27), we have, for $0<m<2$

$$\int_0^1 y^2 dx \leqq \frac{(3-m)}{3(2-m)} \int_0^1 [x^m (y')^2 + m x^{m-2} y^2] dx. \tag{29}$$

Equality in (27) may be achieved at the point t by choosing $y_t(x) = w(x;\, t)$ as given in (26). If $0<m\leqq 1$ then the equality in (28) is achieved at $t=1$ when $y(x)=x$ for all x.

References

1. Hardy, Littlewood and Polya, *Inequalities*, Cambridge University Press, 1934.

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