PRINCIPAL REPRESENTATIONS IN COMMUTATIVE SEMIGROUPS

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1. **Introduction.** This paper is intended as a note to [2], and the definitions and notations of that paper will be used throughout, except that $S: T$, where S is a co-ideal and T is the null set will be defined to be S, rather than $\mathfrak{M}$. This change will not affect the results of [2]. To avoid repetition, let the terms "semigroup" and "ring" always signify "commutative semigroup" and "commutative ring," respectively. As in [2], let $\mathfrak{M}$ denote a semigroup and $\mathfrak{F}$ a family of its co-ideals. Let S and T denote arbitrary elements of $\mathfrak{F}$ and let P and Q denote, respectively, prime and primary elements of $\mathfrak{F}$. Let M denote a p-set contained in $\mathfrak{M}$.

The purpose of [2] was to reproduce, in so far as possible, the ideal theory of rings in semigroups in such a way that ring ideal theory could be obtained as a special case by considering those semigroups which were actually rings. To that end, co-ideals were defined in such a way that the complement of an ideal of a ring, R, would be a co-ideal of the multiplicative semigroup of R. In order that those co-ideals which were not the complements of ring ideals could be discarded when desired, [2] considered only those co-ideals which were members of a particular family, $\mathfrak{F}$, of co-ideals of a semi-group, $\mathfrak{M}$, satisfying

I. $S \in F$ implies $S^{\blacktriangle} \in \mathfrak{F}$.

II. The arbitrary union of $\mathfrak{F}$-elements is an $\mathfrak{F}$-element.

III. $S \in \mathfrak{F}$ and M a nonempty p-set contained in S, implies that $S: M \in \mathfrak{F}$.

The complements of ring ideals obviously are an example of such a family. If T is any subset of $\mathfrak{M}$, let CT denote the complement of T in $\mathfrak{M}$.

The chief results of [2] were the decompositions of every proper member of a family, $\mathfrak{F}$, into a finite union of primary members of $\mathfrak{F}$ and into a finite union of proper, pairwise separate, members of $\mathfrak{F}$, under certain further restrictions on $\mathfrak{F}$. Here another kind of a decomposition will be studied and another ring ideal theory result reproduced in semigroup co-ideal theory.

2. **Preliminary definitions.**

Definition 2.1. If T, $S \in \mathfrak{F}$, T will be called *prime to S* if there

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exists $a \notin T$ such that $a \cdot S \subset S$.

If $\mathfrak{M}$ is a ring and if CS and CT are ideals, it is seen that T prime to S is equivalent to CT prime to CS in the sense of [**1**, 16], but that this primeness is stronger than that of [**3**, 25].

Obviously T is nonprime to S if and only if $T \supset S: S$.

It will now be necessary to further restrict $\mathfrak{F}$ by:

IV. Every nonempty set of elements of $\mathfrak{F}$ contains a minimal element.

V. If $a \in S \in \mathfrak{F}$, then $S: \{a\} \in \mathfrak{F}$.

It should be pointed out that if $\mathfrak{M}$ is a ring and $\mathfrak{F}$ is the collection of the complements of the ring ideals, condition V is always satisfied and IV is equivalent to the ascending chain condition.

If $S \in \mathfrak{F}$, consider the totality of elements of F which contain $S: S$. This collection is nonempty, as it contains S, so by IV, there is a $T \in \mathfrak{F}$ such that T is minimal nonprime to S.

THEOREM 2.1. *If T is minimal nonprime to S, then T is prime.*

PROOF. If $S: S$ is non-null, $\mathfrak{F}$ contains a prime element, P, such that $T \supset P \supset S: S$, by the first theorem of [**2**], since $S:S$ is a p-set. Since T is minimal, $T = P$.

If $S:S$ is null and $T^{\blacktriangle}$ is non-null, then T still contains a prime member of $\mathfrak{F}$ by the same theorem and this prime member of $\mathfrak{F}$ is equal to T by the minimal property. If $T^{\blacktriangle}$ is null, then T minimal implies $T = T^{\blacktriangle} = S:S$ is null, so T is a prime member of $\mathfrak{F}$.

By IV, if T is any co-ideal of $\mathfrak{F}$ which is nonprime to S, then $T \supset P \supset S:S$, where P is minimal with this property.

DEFINITION 2.2. If P is minimal nonprime to S, then $S:P$ is called a *principal component* of S.

N.B. If P is non-null and $S:P$ is non-null, then $S:P$ is an example of a *segregated component* of [**2**].

3. Decomposition theorems.

THEOREM 3.1. *Every element of $\mathfrak{F}$ is the union of its principal components.*

PROOF. If $S \in \mathfrak{F}$, let $\{P_\alpha\}$ be all the members of $\mathfrak{F}$ which are minimal nonprime to S. If any $P_\alpha \not\subset S$, then $S:P_\alpha$ is null, so we may suppose that each $P_\alpha \subset S$. Since S contains each $S:P_\alpha$, $S \supset \bigcup_\alpha S:P_\alpha$. Now suppose that $a \in S$. By V, $S:S \subset S: \{a\} \in \mathfrak{F}$. Since $S: \{a\}$ is nonprime to S, it contains one of the P_α. But $S: \{a\} \supset P_\alpha$ implies that $a \in S:P_\alpha$ implies that $a \in \bigcup_\alpha S:P_\alpha$, so $S = \bigcup_\alpha S:P_\alpha$.

DEFINITION 3.1. $S = \bigcup_\alpha S:P_\alpha$ is called the *principal representation or principal decomposition* of S.

This union may be finite or arbitrary. The principal representation gives less information about S than it does about $\mathfrak{F}$. The property of a co-ideal being primary or the union of primary co-ideals is not influenced by increasing the size of $\mathfrak{F}$. However, the principal representation may become very much shorter if the size of the family is increased. In fact, if $S:S\in\mathfrak{F}$, the principal representation reduces to $S=S:(S:S)$. Conversely, if P minimal nonprime to S exists such that $S=S:P$, then $P=S:S$, so $S:S\in\mathfrak{F}$. If Q is primary, then $Q:Q=Q^{\blacktriangle}\in\mathfrak{F}$, so $Q=Q:Q^{\blacktriangle}$ is the principal representation of Q.

Restriction V on the family will now be dropped.

THEOREM 3.2. *If $S=\bigcup_\alpha Q_\alpha$, where the Q_α are primary members of $\mathfrak{F}$, then S is the union of its principal components. Further, if $\bigcup_\alpha Q_\alpha$ is a finite union, S has only finitely many distinct principal components.*

PROOF. We may suppose now that for all β, $Q_\beta\not\subset\bigcup_{\alpha\neq\beta} Q_\alpha$ and that the $Q_\alpha^{\blacktriangle}$ are distinct. Compare [2, §4].

CONTENTION. *Each Q_α is nonprime to S.* If Q_β is prime to S, then there exists $a\notin Q_\beta$ such that $aQ_\beta\subset S$. Now aQ_β is disjoint from Q_β, so $aQ_\beta\subset\bigcup_{\alpha\neq\beta} Q_\alpha$, which implies that $Q_\beta\subset\bigcup_{\alpha\neq\beta} Q_\alpha$, contradicting the above assumption.

So each Q_α contains P_α, where P_α is minimal nonprime to S. Actually $P_\alpha\subset Q_\alpha^{\blacktriangle}$, so since Q_α is primary, $P_\alpha Q_\alpha\subset Q_\alpha^{\blacktriangle}\cdot Q_\alpha\subset Q_\alpha\subset S$, i.e. $Q_\alpha\subset S:P_\alpha$. If the $\{P_\gamma\}$ are all of the minimal elements of $\mathfrak{F}$ containing $S:S$, certainly the $\{P_\alpha\}$ are among them. We have $S=\bigcup_\alpha Q_\alpha \subset\bigcup_\alpha S:P_\alpha\subset\bigcup_\gamma S:P_\gamma$. On the other hand $\bigcup_\gamma S:P_\gamma\subset S$, as before.

Suppose now that $S=Q_1\cup\cdots\cup Q_n$ is a normed union. See [2, §4]. Suppose that P is minimal nonprime to S. If P is null, then $S:P=S$ is the only principal component and the result is proved. Otherwise the fourth theorem of [2] shows that $S:P$ equals the union of those Q_i which contain P. In fact, $S:P=S:M$, where M is the intersection of the co-radicals of those Q_i which contain P. For, again using [2, Theorem 4], $S:M$ is the union of those Q_i which contain M and, as each Q_i which contains P also contains M, $S:P\subset S:M$. On the other hand, $P\subset Q_i$ implies that $P\subset Q_i^{\blacktriangle}$, so that $P\subset\bigcap_{Q_i\supset P} Q_i^{\blacktriangle}=M$, so $S:P\supset S:M$. Thus the number of possible distinct principal components of S is bounded by the number of such M's, i.e. by the number of non-null subsets of the n Q_i, or by 2^n-1.

4. An application. The principal decomposition theory has a particularly interesting application to single-primed families, i.e. those in which every prime element is maximal. In this case [2, Lemma 1] shows that every principal component is a primary element of the family. Thus, by Theorem 3.1, every member of a single-primed family, satisfying assumptions I through V, is a union of primary members of the family. For this special case the result is achieved without making assumptions (a) and (b) of [**2**, §7].

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A LIMIT THEOREM FOR PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

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I. **Introduction.** Let $[X(t);\ 0\leqq t\leqq 1]$ be the Wiener process. That is, a Gaussian process of real valued variables with $E[X(t)]=0$ and $E[X(s)X(t)]=\min(s,t)$. Levy [**1**] showed that with probability one

$$\lim_{N\to\infty}\sum_{k=1}^{N}[X(t_k)-X(t_{k-1})]^2=1 \qquad \text{for } 0=t_0<t_1<\cdots<t_N=1.$$

Cameron and Martin [**2**] showed this independently for $t_k=k/2^N$. Baxter [**3**] recently extended these results by the following theorem:

Theorem. *If $[X(t);\ 0\leqq t\leqq 1]$ is a Gaussian process satisfying $E[X(t)]=m(t)$ and $E[X(s)X(t)]-m(s)m(t)=r(s,t)$, $r(s,t)$ is continuous in $0\leqq s,t\leqq 1$ and has uniformly bounded second derivatives for $s\neq t$.*

Let

$$D^+(t)=\lim_{s\to t^+}\frac{r(t,t)-r(s,t)}{t-s},$$

$$D^-(t)=\lim_{s\to t-}\frac{r(t,t)-r(s,t)}{t-s},$$

and

$$f(t)=D^-(t)-D^+(t).$$

Then with probability one

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