ON SOME MERCERIAN THEOREMS IN SUMMABILITY

M. R. PARAMESWARAN

1. **Introduction.** Mercer's Theorem [**10**; **4**, Theorem 51] and various extensions of it have been treated in many recent papers [**1**; **3**; **7**; **8**] etc. It is the object of this paper to use functional analysis to prove Mercerian Theorems for ordinary and absolute summability; we prove also a few results on absolute summability that may have some independent interest. Functional analysis treatment of Mercerian theorems for ordinary summability has also been given by Širkov [**13**].[1]

2. **Some notations and lemmas.** We note the following definitions and lemmas, some of which are quite well known. We shall use the symbols (c_0), (c) and (m) to denote respectively the set of all sequences converging to zero, the set of all convergent sequences, and the set of all bounded sequences. The sequence-to-sequence transformation given by the equations

$$y_n = \sum_{k=0}^{\infty} a_{nk}x_k, \qquad (n = 0, 1, 2, \cdots)$$

will be written as a matrix equation $y=Ax$ where y and x are column vectors, $y=\{y_n\}$ and $x=\{x_n\}$. Let $(\mathfrak{M})$ and $(\mathfrak{N})$ be given sets of sequences. Then $\Gamma(\mathfrak{M}, \mathfrak{N})$ will denote the set of matrices A such that $x\in(\mathfrak{M})$ implies $Ax\in(\mathfrak{N})$. As we shall see, it will be convenient to work with this notation.

LEMMA 1 (Hille [**5**, p. 92]). *If Γ is any complex Banach algebra with unit element I, then for every element $A\in\Gamma$ the element $I+qA$ where q is any complex number such that $|q|\cdot\|A\|<1$, has an inverse in Γ.*

LEMMA 2. *The sets of matrices $\Gamma(c_0, c_0)$, $\Gamma(c, c)$ and $\Gamma(m, m)$ are all complex Banach algebras, with the norm in each case being defined by: $\|A\| = \text{l.u.b.}_{0\leq n<\infty} \sum_{k=0}^{\infty} |a_{nk}|$.*

A proof for $\Gamma(c, c)$ is found in the author's paper [**11**]; the other cases are similarly proved.

LEMMA 3. (Zeller [**14**]). *Let $A\in\Gamma(\mathfrak{M}, \mathfrak{M})$, $(\mathfrak{M})=(c_0)$ or (c). If $Ax\in(\mathfrak{M})$ implies $x\in(m)$, then $x\in(\mathfrak{M})$.*

Received by the editors August 23, 1956 and, in revised form, March 4, 1957.

[1] The author is thankful to the referee for this and other valuable remarks.

LEMMA 4. *Let* $A\in\Gamma(\mathfrak{M}, \mathfrak{M})$, $(\mathfrak{M})=(c_0)$ *or* (c). *If the matrix* B *is such that* $AB=BA=I$ *and* $B\in\Gamma(m, m)$ *then* $B\in\Gamma(\mathfrak{M}, \mathfrak{M})$.

The case $(\mathfrak{M})=(c)$ is proved elsewhere [**12**]; the other case is proved similarly.

3. Some Mercerian theorems.

We now give two theorems, the first of which includes, and slightly generalizes, Theorems 1, 2, 3 and 4 of Love [**8**], and is itself included in the second theorem the results of which were first proved by Agnew (see [**1**; **2**]).

THEOREM 1. *Let* $A\in\Gamma(\mathfrak{M}, \mathfrak{M})$ *where* $(\mathfrak{M})=(c_0)$, (c) *or* (m). *Then the conditions* $(I+qA)x\in(\mathfrak{M})$,

$$(1) \qquad |q|\cdot\limsup_{n\to\infty}\sum_{k=0}^{\infty}|a_{nk}|<1$$

imply that $x\in(\mathfrak{M})$ *provided that either* (i) $x\in(m)$ *or* (ii) A *is a lower-semimatrix (i.e.,* $a_{nk}=0$ *for* $k>n$).

THEOREM 2. *Let* $A\in\Gamma(\mathfrak{M}, \mathfrak{M})$, $(\mathfrak{M})=(c_0)$ *or* (c) *or* (m) *and let*

$$(2) \qquad \liminf_{n\to\infty}\Big\{|a_{nn}|-\sum_{k\neq n}|a_{nk}|\Big\}>\lambda>0;$$

then $Ax\in(\mathfrak{M})$ *implies that* $x\in(\mathfrak{M})$, *provided that either* (i) $x\in(m)$ *or* (ii) A *is a lower-semimatrix.*

PROOF. We may assume without loss of generality that $|a_{nn}|-\sum_{k\neq n}|a_{nk}|>\lambda>0$ for all n, for we may alter a finite number of rows of the matrix A without affecting its summability properties. Define the matrix B by $I+B\equiv(a_{nk}/a_{nn})$. Since $\|B\|<1$ by (2), we have by Lemmas 1 and 2 that $C=(I+B)^{-1}\in\Gamma(\mathfrak{M}, \mathfrak{M})$. Also $I+B=PA$ where $P=(p_{nk})$ is a diagonal matrix with $p_{nn}=1/a_{nn}$ for all n. Since $\|A\|\geqq|a_{nn}|>\lambda>0$, both P and P^{-1} belong to $\Gamma(m, m)$. Since further $PAC=CPA=I=P^{-1}(PAC)P=ACP$ we see that $A^{-1}=CP\in\Gamma(m,m)$. Therefore, if A is a lower-semimatrix then $Ax\in(\mathfrak{M})$ implies $x\in(m)$, and it follows from either of Lemmas 3, 4 that $x\in(\mathfrak{M})$. This proves part (ii) of the theorem.

If $x\in(m)$ and $Ax\in(\mathfrak{M})$, then by what has been proved above and Lemma 4, we see that $A^{-1}\in\Gamma(\mathfrak{M}, \mathfrak{M})$ and part (i) of the theorem is also proved.

4. Absolute summability.

Corresponding to the Mercerian theorems for ordinary summability, there are also analogues for absolute summability—where convergence is replaced by absolute convergence. The sequence $x=\{x_n\}$ is said to be absolutely convergent if and only if the series $\sum_{n=0}^{\infty}(x_n - x_{n-1})$ is absolutely convergent (x_{-1} is taken to be zero). We shall denote the set of all absolutely convergent sequences by $(|c|)$, and the set of all sequences $x=\{x_n\}$ such that $\sum_{n=0}^{\infty}|x_n| < \infty$, by the symbol (l_1). Theorem 3 below is the analogue of Lemma 2 for absolute summability; we shall prove it by a somewhat indirect method by a number of easy steps, proving a few lemmas which may also be considered to be of some independent interest.

LEMMA 5. (See Mears [9]). *The matrix* $P=(p_{nk})$, $(n, k=0, 1, 2, \cdots)$ *belongs to* $\Gamma(|c|, |c|)$, *that is,* $Px\in(|c|)$ *for every* $x\in(|c|)$ *if and only if*

$$(3) \qquad \|P\|^* \equiv \underset{0\leqq k<\infty}{\text{l.u.b.}} \sum_{n=0}^{\infty}\left|\sum_{k=h}^{\infty}(p_{nk} - p_{n-1,k})\right| < \infty$$

where $p_{-1,k}=0$ *and*

$$g_n = \sum_{k=0}^{\infty} p_{nk} \qquad \textit{exists for each } n = 0, 1, 2, \cdots.$$

LEMMA 6. (See Knopp and Lorentz [6]). *The matrix* $A=(a_{nk})$ *belongs to* $\Gamma(l_1, l_1)$ *if and only if*

$$(4) \qquad \|A'\| \equiv \underset{0\leqq k<\infty}{\text{l.u.b.}} \sum_{n=0}^{\infty} |a_{nk}| < \infty$$

(*or equivalently, if and only if* $A'\in\Gamma(m, m)$ *where* A' *is the transpose of the matrix* A, *so that obviously* $\|A\|'=\|A'\|$).

LEMMA 7. *Let* $A=(a_{nk})$ *and let* $G=(g_{nk})$ *be defined by*

$$(5) \qquad g_{nk} = a_{0k} + a_{1k} + \cdots + a_{nk} \qquad (n, k = 0, 1, 2, \cdots).$$

Then $A\in\Gamma(l_1, l_1)$ *if and only if* $G\in\Gamma(l_1, |c|)$.

The proof is immediate from the equation

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty} a_{nk}u_k\right| = \sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}(g_{nk} - g_{n-1,k})u_k\right|,$$

for the equation (5) is equivalent to the relation

$$(6) \qquad a_{nk} = g_{nk} - g_{n-1,k} \qquad (n, k = 0, 1, 2, \cdots).$$

DEFINITION. The set of matrices $A\in\Gamma(\mathfrak{M}, \mathfrak{N})$ such that $\lim_{k\to\infty} a_{nk} = 0$ for each $n=0, 1, 2, \cdots$ will be denoted by $\Gamma_0(\mathfrak{M}, \mathfrak{N})$.

LEMMA 8. *Let the matrices* A *and* G *be related as above. Then,* $A\in\Gamma_0(l_1, l_1)$ *if and only if* $G\in\Gamma_0(l_1, |c|)$.

The proof is immediate from equations (5) and (6).

LEMMA 9. *The sets of matrices* $\Gamma(l_1, l_1)$ *and* $\Gamma_0(l_1, l_1)$ *are complex Banach algebras under the norm defined by* (4).

This is a consequence of the fact that $\Gamma(l_1, l_1)$ and $\Gamma_0(l_1, l_1)$ are, under the correspondence $A \rightarrow A'$, isometric with the Banach algebras (see Lemma 2) $\Gamma(m, m)$ and $\Gamma(c_0, c_0)$ respectively.

It is easily verified that the following result is true:

LEMMA 10. *Let* $P=(p_{nk}) \in \Gamma(|c|, |c|)$ *and let*

$$g_{nk} = p_{nk} + p_{n,k+1} + \cdots + \cdots \qquad (n, k = 0, 1, 2, \cdots). \tag{7}$$

Then $G=(g_{nk}) \in \Gamma_0(l_1, |c|)$ *and*

$$p_{nk} = g_{nk} - g_{n,k+1} \qquad (n, k = 0, 1, 2, \cdots). \tag{8}$$

The converse is also true.

REMARK. The statements $A \in \Gamma_0(l_1, l_1)$, $G \in \Gamma_0(l_1, |c|)$ and $P \in \Gamma(|c|, |c|)$ are equivalent. For, $s=\{s_n\} \in (l_1)$ if and only if $\sigma=\{\sigma_n\} \in (|c|)$, where $\sigma_n = s_0+s_1+\cdots+s_n$. It is easy to verify that if $A \in \Gamma_0(l_1, l_1)$ and $s \in (l_1)$, then $As=t=\{t_n\} \in (l_1)$, $Gs=\tau \in (|c|)$ where $\tau=\{\tau_n\}$, $\tau_n = t_0+t_1+\cdots+t_n$ and further that $P\sigma=\tau$. The correspondence $A \leftrightarrow G \leftrightarrow P$ is thus the "natural" one.

We have established above one-to-one correspondences between $\Gamma_0(l_1, l_1)$ and $\Gamma_0(l_1, |c|)$ on the one hand and between $\Gamma_0(l_1, |c|)$ and $\Gamma(|c|, |c|)$ on the other. Thus we have the

LEMMA 11. *The correspondences* $A \leftrightarrow G \leftrightarrow P$ *where* $A \in \Gamma_0(l_1,l_1)$, $G \in \Gamma_0(l_1, |c|)$ *and* $P \in \Gamma(|c|,|c|)$, *as defined above, are one-to-one; and the correspondence* $A \leftrightarrow P$ *is expressed by either of the equivalent formulae*

$$
\begin{aligned}
p_{nk} &= \sum_{i=0}^{\infty} (a_{ik} - a_{i,k+1}) \quad (n, k = 0, 1, 2, \cdots), \\
a_{nk} &= \sum_{i=k}^{\infty} (p_{ni} - p_{n-1,i}) \quad (n, k = 0, 1, 2, \cdots).
\end{aligned} \tag{9}
$$

The equations giving (p_{nk}) and (a_{nk}) in terms of each other are verified by simple calculation.

LEMMA 12. *Let the matrix* G *be related to* $A \in \Gamma(l_1, l_1)$ *by the equation* (5) *and let* $B \in \Gamma(l_1, l_1)$. *Then* $GB \in \Gamma(l_1, |c|)$ *and is similarly related to* $AB \in \Gamma(l_1, l_1)$.

PROOF. The equations (4) and (6) give

$$|g_{ni}| \leqq \sum_{k=0}^{\infty} |g_{ki} - g_{k-1,i}| \leqq \|A\|' < \infty;$$

also, $\sum_{i=0}^{\infty} |b_{ik}| \leqq \|B\|' < \infty$. Therefore the sums $(GB)_{nk} \equiv \sum_{i=0}^{\infty} g_{ni}b_{ik}$ exist for $n, k=0, 1, 2, \cdots$; and hence the product GB also exists.

Now, $(AB)_{nk} = \sum_{i=0}^{\infty} a_{ni}b_{ik} = \sum_{i=0}^{\infty} (g_{ni} - g_{n-1,i})b_{ik} = (GB)_{nk} - (GB)_{n-1,k}$ and the result follows from the equivalence of the relations (5) and (6).

LEMMA 13. *The one-to-one correspondence* $A \leftrightarrow P$ *defined above between* $\Gamma_0(l_1, l_1)$ *and* $\Gamma(|c|, |c|)$ *is an isomorphism.*

PROOF. Let $A, B \in \Gamma_0(l_1, l_1)$ and let the matrices corresponding to them in $\Gamma_0(l_1, |c|)$, $\Gamma(|c|, |c|)$ be G, H and P, Q respectively; that is, $A \leftrightarrow G \leftrightarrow P$, $B \leftrightarrow H \leftrightarrow Q$ and $G, H \in \Gamma_0(l_1, |c|)$; $P, Q \in \Gamma(|c|, |c|)$.

It is obvious that $A+B \leftrightarrow P+Q \in \Gamma(|c|, |c|)$, $\lambda A \leftrightarrow \lambda P$ where λ is any complex constant. [By Lemma 9 we have that $A+B$, λA and AB all belong to $\Gamma_0(l_1, l_1)$.] We have to prove that $PQ \in \Gamma(|c|, |c|)$ and that $AB \leftrightarrow PQ$.

In view of Lemmas 10, 11 and 12 it is enough to prove that the product PQ exists and that

$$(10) \qquad (PQ)_{nk} = (GB)_{nk} - (GB)_{n,k+1} \qquad (n, k = 0, 1, 2, \cdots).$$

Now,

$$\begin{aligned}
(GB)_{nk} - (GB)_{n,k+1} &= \sum_{i=0}^{\infty} g_{ni}(b_{ik} - b_{i,k+1}) \\
&= \sum_{i=0}^{\infty} g_{ni}(h_{ik} - h_{i-1,k} - h_{i,k+1} + h_{i-1,k+1}) \text{ by (6)}, \\
&= \sum_{i=0}^{\infty} g_{ni}\{(h_{ik} - h_{i,k+1}) + (h_{i-1,k+1} - h_{i-1,k})\} \\
&= \sum_{i=0}^{\infty} (g_{ni} - g_{n,i+1})(h_{ik} - h_{i,k-1}) \\
&= \sum_{i=0}^{\infty} p_{ni}q_{ik}.
\end{aligned}$$

This establishes that the product PQ exists and that it satisfies the relation (10). Since $AB \leftrightarrow GB \in \Gamma_0(l_1, |c|)$ and $GB \leftrightarrow PQ \in \Gamma(|c|, |c|)$, the lemma is proved.

COROLLARY. *The sets* $\Gamma(|c|, |c|)$ *and* $\Gamma(c_0, c_0)$ *are isomorphic under the correspondence* $P \leftrightarrow A'$ *where* A' *is the transpose of* A.

THEOREM 3. *The set* $\Gamma(|c|, |c|)$ *is a complex Banach algebra where the norm of* $P \in \Gamma(|c|, |c|)$ *is defined by* $\|P\|^*$ *given in* (3).

PROOF. We have from the relation (3), $\|P\|^* = \|A\|' = \|A'\|$. Also, it is easily verified that the equation (3) defines a norm over $\Gamma(|c|, |c|)$. The theorem follows then from Lemma 2 and the corollary above, in view of the isometry.

Theorem 3 above leads us to the following generalization of Bosanquet's analogue [**3**], for absolute summability of Mercer's theorem; it is the analogue, for absolute summability, of Theorem 2.

THEOREM 4. *If* $P \in \Gamma(|c|, |c|)$ *is a lower-semimatrix and*

$$|p_{nn}| - \sum_{k=n+1}^{\infty} \left| \sum_{i=n}^{k} (p_{ki} - p_{k-1,i}) \right| > \lambda > 0$$

for all $n = 0, 1, 2, \cdots$, *then* $Px \in (|c|)$ *implies* $x \in (|c|)$.

PROOF. Let $B = A' \in \Gamma(c_0, c_0)$ correspond to P under the isomorphism stated in the corollary to Lemma 13. Then it is easily verified that B satisfies the condition $|b_{nn}| - \sum_{k=n+1}^{\infty} |b_{nk}| > \lambda > 0$ $(n = 0, 1, 2, \cdots)$ and hence, as seen in the proof of Theorem 2, we have that $B^{-1} \in \Gamma(c_0, c_0)$. It follows now from the isomorphism that $P^{-1} \in \Gamma(|c|, |c|)$. Also, P and P^{-1} are lower-semimatrices and therefore if $Px \in (|c|)$ then $x = (P^{-1}P)x = P^{-1}(Px) \in (|c|)$ and the theorem is proved.

It is easily seen that the Mercerian theorem for absolute summability given by Love [**8**, Theorem 5] is an immediate corollary of Theorem 4 proved above. It can also be proved that the following result for Hausdorff matrices, which includes and is more general than the results given by Love [**8**, Corollaries 4, 5], is also a corollary of our Theorem 4. We give an alternative short and interesting proof of the result.

THEOREM 5. *If* $A \in \Gamma(c, c)$ *is a Hausdorff matrix and satisfies the condition*

$$|a_{nn}| - \sum_{k=0}^{n-1} |a_{nk}| > \lambda > 0 \quad (n = 0, 1, 2, \cdots) \tag{11}$$

then $Ax \in (|c|)$ *implies that* $x \in (|c|)$.

PROOF. As shown in the proof of Theorem 2, we have now

$A^{-1} \in \Gamma(c, c)$, and since A is Hausdorff, so is A^{-1}. Now, Knopp and Lorentz [6, Theorem 3] have proved that any Hausdorff matrix belonging to $\Gamma(c, c)$ belongs also to $\Gamma(|c|, |c|)$. Thus $A^{-1} \in \Gamma(|c|, |c|)$ and the theorem is immediately proved.

References

1. R. P. Agnew, *Mercer's summability theorem*, J. London Math. Soc. vol. 29 (1954) pp. 123–125.

2. ———, *Equivalence of methods for evaluation of sequences*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 550–556.

3. L. S. Bosanquet, *An analogue of Mercer's theorem*, J. London Math. Soc. vol. 13 (1938) pp. 177–180.

4. G. H. Hardy, *Divergent series*, Oxford, 1949.

5. E. Hille, *Functional analysis and semigroups*, New York, 1948.

6. K. Knopp and G. Lorentz, *Beitrage zur absoluten Limitierung*, Archiv der Mathematik vol. 2 (1949–1950) pp. 10–16.

7. R. T. Leslie and E. R. Love, *An extension of Mercer's theorem*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 448–457.

8. E. R. Love, *Mercer's summability theorem*, J. London Math. Soc. vol. 27 (1952) pp. 413–429.

9. F. M. Mears, *Absolute regularity and the Nörlund mean*, Ann. of Math. (2) vol. 38 (1937) pp. 594–601.

10. J. Mercer, *On the limits of real variants*, Proc. London Math. Soc. (2) vol. 5 (1907) pp. 206–224.

11. M. R. Parameswaran, *Some converse theorems on summability*, Proceedings of the Indian Academy of Sciences (Sec. A) vol. 36 (1952) pp. 363–369.

12. ———, *On the reciprocal of a K-matrix*, J. Indian Math. Soc. vol. 20 (1956) pp. 329–331.

13. F. V. Širkov, *O teoreme Mercera*, Uspehi Matematičeskih Nauk. (NS) vol. 10 (1955) pp. 167–170.

14. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Zeit. vol. 53 (1951) pp. 463–487.

Ramanujan Institute of Mathematics, Madras