# A THEOREM ON CONTINUED FRACTIONS

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1. **Introduction.** Let $[a_1, \cdots, a_n]$ denote the simple continued fraction whose successive quotients $a_1, \cdots, a_n$ are elements of a field $K$. Following the procedure of Milne-Thompson,[1] define the formal numerator $P$ and formal denominator $Q$ of this continued fraction by means of

$$(1) \qquad \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}.$$

For brevity, we write $[a_1, \cdots, a_n] \sim P/Q$.

Now let $K$ be a skew-field, and let $R = K[x]$ be the ring of polynomials in an indeterminate $x$ with coefficients in $K$, where we assume that $x$ commutes with all elements of $K$. For $f_1, \cdots, f_n \in R$, define $[f_1, \cdots, f_n] \sim P/Q$ where (as above)

$$(2) \qquad \begin{pmatrix} f_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} f_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Next let $f \rightarrow f^*$ denote any homomorphism of $(R, +)$ into itself which leaves $K$ elementwise fixed, and satisfies $(af)^* = af^*$ for all $a \in K, f \in R$. We shall prove:

THEOREM. *If $f_1, \cdots, f_k \in R$ are such that $[f_1, \cdots, f_k] \sim P/Q$ where $P, Q \in K$, then also $[f_1^*, \cdots, f_k^*] \sim P/Q$.*

Thus, any identity of the form $[f_1, \cdots, f_k] \sim P/Q$, $P, Q \in K$, depends only upon the additive structure of $R$, and not upon its multiplicative structure. This result will be basic in a future paper on the automorphisms of $GL_2(R)$.[2]

2. Several lemmas will be needed for the proof of the theorem.

LEMMA 1. Let *$[f_1, \cdots, f_n] \sim P/Q$, where $f_1, \cdots, f_n \in R$. Suppose that $f_r, f_{r+1}, \cdots, f_s$ $(s \geqq r)$ satisfy*

$$(3) \qquad f_r, \cdots, f_s \in K, \quad \begin{pmatrix} f_r & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} f_s & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c \\ b & d \end{pmatrix}.$$

*Set $e = dc^{-1}$, $f'_{s+1} = bf_{s+1}c^{-1}$, $f'_{s+2} = cf_{s+2}b^{-1}$, $\cdots$. Then*

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[1] Proceedings of the Edinburgh Mathematical Society, 1933.

[2] I. Reiner, *A new type of automorphism of the general linear group over a ring*, to appear in Annals of Mathematics.

$$\begin{pmatrix} f_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_{r-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{r-1} + e + f'_{s+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f'_{s+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \tag{4}$$
$$\begin{pmatrix} f'_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix},$$

*where $t$ is either $b$ or $c$, depending on the parity of $n-s$.*

PROOF. From (2) we have

$$\begin{pmatrix} P \\ Q \end{pmatrix}$$
$$= \begin{pmatrix} f_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_{r-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & e \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} f_{s+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} f_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_{r-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & e \end{pmatrix} \begin{pmatrix} f'_{s+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f'_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix},$$

which readily implies (4).

LEMMA 2. *Let $f_1, \cdots, f_n \in R$, $f_1 \notin K$, $f_n \notin K$, $n>1$. Define $[f_i, \cdots, f_n] \sim P_i/Q_i$, $1 \leqq i \leqq n$. If both $P_1$ and $Q_1$ lie in $K$, then there exists a run of consecutive elements $f_r, \cdots, f_s$ $(1<r\leqq s<n)$ such that* (3) *holds.*

PROOF. Suppose that for every run of consecutive elements $f_r, \cdots, f_s$ which lie in $K$, we have $[f_r, \cdots, f_s] \sim a/b$ with $a \neq 0$. We shall obtain a contradiction. Using (2), we have

$$P_i = f_i P_{i+1} + Q_{i+1}, \qquad Q_i = P_{i+1}, \qquad 1 \leqq i \leqq n-1.$$

Consequently

(5) If $\deg P_i \leqq \deg Q_i$, and if $f_i \notin K$, then $\deg P_{i+1} < \deg Q_{i+1}$.

Since $\deg P_1 \leqq \deg Q_1$, and $f_1 \notin K$, (5) shows that $\deg P_2 < \deg Q_2$. If also $f_2 \notin K$, we have likewise: $\deg P_3 < \deg Q_3$. We may continue this process until we reach the first of the elements $f_1, f_2, \cdots$ which lies in $K$, say $f_r$; then $\deg P_r < \deg Q_r$. Let $s$ be chosen so that $f_r, \cdots, f_s \in K$, $f_{s+1} \notin K$. By the supposition at the beginning of the proof, we have $[f_r, \cdots, f_s] \sim a/b$ with $a \neq 0$. Setting

$$\begin{pmatrix} f_r & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_s & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

we have at once

$$P_r = aP_{s+1} + cQ_{s+1}, \qquad Q_r = bP_{s+1} + dQ_{s+1}.$$

Since $\deg P_r < \deg Q_r$, surely $\deg P_{s+1} > \deg Q_{s+1}$ is impossible. Hence $\deg P_{s+1} \leqq \deg Q_{s+1}$.

We may therefore continue this process of descent, eventually obtaining $\deg P_n \leqq \deg Q_n$. However, $P_n = f_n$, $Q_n = 1$, and $f_n \notin K$, so we have reached a contradiction. This completes the proof of the lemma.

We now prove the theorem stated in §1, by induction on $k$. The theorem obviously holds for $k=1$. Suppose it proved for $1 \leqq k \leqq n-1$, where $n \geqq 2$, and suppose that $[f_1, \cdots, f_n] \sim P/Q$, where both $P$ and $Q$ lie in $K$; we shall show that $[f_1^*, \cdots, f_n^*] \sim P/Q$.

To begin with, assume that $f_1 \in K$. Then (2) implies

$$\begin{pmatrix} f_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -f_1 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}.$$

By the induction hypothesis this remains valid when each $f_i$ $(1 \leqq i \leqq n)$ is replaced by $f_i^*$; hence (2) also remains valid under this replacement.

Secondly, assume that $f_n \in K$. Since the result follows trivially from the induction hypothesis when $f_n = 0$, we need only consider the case where $f_n \neq 0$. Equation (2) may then be written as

$$\begin{pmatrix} P \\ f_n^{-1} Q \end{pmatrix} = \begin{pmatrix} f_1 f_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_n^{-1} f_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_n^{-1} f_{n-2} & 1 \\ 1 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} f_{n-1} f_n + 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

provided that $n$ is even, with an analogous result for odd $n$. By the induction hypothesis, this remains valid under the substitution $f \to f^*$, which again implies the desired result.

We are thus left with the case where $n \geqq 2$, $f_1 \notin K$, $f_n \notin K$, and $[f_1, \cdots, f_n] \sim P/Q$ with both $P$ and $Q$ in $K$. By Lemma 2, there exists a run of consecutive elements $f_r, \cdots, f_s$ $(s \geqq r)$ satisfying (3). Keeping the notation of Lemma 1, equation (2) implies the validity of (4), with $t \neq 0$. Upon multiplying the successive quotients $f_n', \cdots, f_1$ in (4) by $t$ or $t^{-1}$ (according to the method used in the preceding paragraph), equation (4) reduces to an equation of type (2). This new equation of type (2) remains true under the substitution $f \to f^*$, as a consequence of the induction hypothesis. Therefore also (4) remains true under $f \to f^*$; as above, we deduce that the original equation (2) remains correct under the substitution $f \to f^*$. This completes the proof of the theorem.

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