ON MONOTONE CONVERGENCE TO SOLUTIONS OF $u' = g(u, t)$

RICHARD BELLMAN

1. **Introduction.** The purpose of this brief note is to show that Newton's method of successive approximation to the solution of a functional equation yields a monotone increasing sequence of approximations to the solution of the first order differential equation,

$$u' = g(u, t), \qquad u(0) = c, \tag{1}$$

provided that $g(u, t)$ is uniformly convex in u for t in some fixed interval $[0, t_0]$.

The connection between the methods and results presented here and the theory of dynamic programming is treated in [1].

2. **Newton's method of approximation.** Write

$$h(u, t) = \frac{\partial g}{\partial u}(u, t), \tag{1}$$

and consider the system of equations

$$\text{(2)} \quad \begin{aligned} &\text{(a)} \quad \frac{du_0}{dt} = g(v_0, t) + (u_0 - v_0)h(v_0, t),\ u_0(0) = c, \\ &\text{(b)} \quad \frac{du_{n+1}}{dt} = g(u_n, t) + (u_{n+1} - u_n)h(u_n, t),\ u_{n+1}(0) = c, \\ &\qquad\qquad n = 0, 1, 2, \cdots, \end{aligned}$$

where $v_0(t)$ is a known function of t which is taken to be continuous over $[0, t_0]$.

This is Newton's method of approximation applied to the differential equation of (1.1).

3. **Montonicity of convergence.** Let us now demonstrate the following result

THEOREM. *Let $h(u, t)$ exist, be continuous, and monotone increasing in u for $0 \leqq t \leqq t_0$. Then*

$$u_0(t) \leqq u_1(t) \leqq \cdots \leqq u_n(t) \leqq \cdots, 0 \leqq t \leqq t_1, \tag{1}$$

where $t_1 > 0$. The limit $u(t) = \lim_{n \to \infty} u_n(t)$ is the solution of (1.1).

PROOF. We begin with the observation that the uniform convexity

Received by the editors March 13, 1957.

of $g(u, t)$ yields the result that

$$g(u, t) = \operatorname*{Max}_{v} [g(v, t) + (u - v)h(v, t)] \tag{2}$$

for $0 \leqq t \leqq t_0$.

Hence

$$\begin{aligned} \frac{du_{n+1}}{dt} &= g(u_n, t) + (u_{n+1} - u_n)h(u_n, t) \\ &\leqq g(v, t) + (u_{n+1} - v)h(v, t), \end{aligned} \tag{3}$$

where $v = v(t)$ is the function which maximizes the function $g(u, t) + (u_{n+1} - u)h(u, t)$. It follows that the solution of

$$\frac{dw}{dt} = g(v, t) + (w - v)h(v, t), \ w(0) = c, \tag{4}$$

majorizes the solution of

$$\frac{du_{n+1}}{dt} = g(u_n, t) + (u_{n+1} - u_n)h(u_n, t), \ u_{n+1}(0) = c, \tag{5}$$

within a common interval of existence, i.e. $w(t) \geqq u_{n+1}(t)$.

Since the function $v(t)$ which maximizes is $u_{n+1}(t)$, we see that the solution of (4) is precisely the function $u_{n+2}(t)$.

This argument showed that $u_1(t) \geqq u_0(t)$ and, inductively, that (3.1) holds within a common interval of existence. That such an interval exists follows the usual lines, and similarly for the proof of convergence which is now equivalent to uniform boundedness of the sequence $\{u_n(t)\}$.

4. **Multi-dimensional case.** The proof presented above hinged on the fact that any function $u(t)$ satisfying the inequality

$$\frac{du}{dt} \leqq a(t)u + b(t), \ u(0) = c, \tag{1}$$

is majorized by the solution of

$$\frac{dv}{dt} = a(t)v + b(t), \ v(0) = c. \tag{2}$$

The corresponding result for systems is not unreservedly true. If $\{x_i(t)\}$ is a set of functions satisfying

$$\frac{dx_i}{dt} \leqq \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t), \ x_i(0) = c_i, \ i = 1, 2, \cdots, n, \tag{3}$$

it is not necessarily true that $x_i(t) \leqq y_i(t)$, $i=1, 2, \cdots, n$, $t \geqq 0$, where the $y_i(t)$ satisfy the equations

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) y_j + b_i(t), \quad y_i(0) = c_i, \qquad i = 1, 2, \cdots, n. \tag{4}$$

If, however, we have

$$a_{ij}(t) \geqq 0, \qquad t \geqq 0, \qquad i \neq j, \tag{5}$$

then the result does hold, cf. [2].

Consequently, if we take the Newton approximations

$$\begin{aligned} \frac{du_{n+1}}{dt} &= f(u_n, v_n) + (u_{n+1} - u_n) \frac{\partial f}{\partial u_n} + (v_{n+1} - v_n) \frac{\partial f}{\partial v_n}, \quad u_{n+1}(0) = c_1, \\ \frac{dv_{n+1}}{dt} &= g(u_n, v_n) + (u_{n+1} - u_n) \frac{\partial g}{\partial u_n} + (v_{n+1} - v_n) \frac{\partial g}{\partial v_n}, \quad v_{n+1}(0) = c_2, \end{aligned} \tag{6}$$

$n=0, 1, 2, \cdots$, with u_0, v_0 prescribed continuous functions, to the system

$$\begin{aligned} \frac{du}{dt} &= f(u, v), \qquad & u(0) = c_1, \\ \frac{dv}{dt} &= g(u, v), \qquad & v(0) = c_2, \end{aligned} \tag{7}$$

we can assert that

$$u_n \leqq u_{n+1}, \quad v_n \leqq v_{n+1}, \quad t \geqq 0, \; n = 0, 1, 2, \cdots, \tag{8}$$

provided that

$$\frac{\partial f}{\partial v} \geqq 0, \qquad \frac{\partial g}{\partial u} \geqq 0, \tag{9}$$

for all u and v, and provided that $f(u, v)$ and $g(u, v)$ are strictly convex functions of u and v.

Bibliography

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