

A NOTE CONCERNING COMPLETELY REGULAR G_δ -SPACES¹

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1. **Introduction.** Let X denote a completely regular topological Hausdorff space. Let νX and βX denote respectively the Hewitt Q -space and the Stone-Čech compactification space associated with X . For the space X , and similarly for other spaces, let $C(X)$ denote the set of all real-valued functions defined and continuous on X . Culminating with the work of Shirota [11], a prolonged series of studies [2; 6; 7; 8; 12] has shown that, for arbitrary completely regular spaces X and Y , an isomorphism of $C(X)$ and $C(Y)$, whether a ring or lattice or multiplicative semigroup isomorphism [5], determines a homeomorphism of νX and νY .

The homeomorphism of νX and νY implies, of course, that νX contains a dense subspace not only homeomorphic to Y , but also such that every function defined and continuous on this subspace may be extended continuously over all of νX . Any dense subset of νX with this extension property will be called an *imbedded* subspace of νX , and, with νX homeomorphic to νY , an imbedded subspace of νX homeomorphic to Y will be called an imbedded image of Y .

Within each space X attention will center at certain points. A G_δ point of a space X is a point p of X such that $\{p\}$ is the intersection of a countable number of open sets of X , or, equivalently, such that $C(X)$ contains a function f with $\{p\} = Z(f)$, where $Z(f)$ indicates the zero set of f in X . An isolated point is a G_δ point, and every non-isolated G_δ point p of X is such that $C(X - \{p\})$ includes a function lacking a continuous extension at p . For the sake of brevity, a point p of X will be called a *generalized G_δ point* if it is either an isolated point of X or is such that $C(X - \{p\})$ contains a function lacking a continuous extension at p . A completely regular space of which each point is a generalized G_δ point will be called a *generalized G_δ space*.

The main purpose of this paper is to describe the imbedded subspaces of νX . First, by example, note is made of the diversity of homeomorphic and imbedded images in νX possible for a single given space Y with $C(Y)$ isomorphic to $C(X)$. Next, the points of νX found in every imbedded subspace are characterized relative both to νX and

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to an arbitrary imbedded subspace of νX . This characterization is then used to distinguish a subspace μX of X which, in certain cases, has a minimal property complementary to the maximal property of νX . The same characterization is also used to extend and unify the known conditions [1; 6; 10] under which the homeomorphism of νX and νY implies the homeomorphism of X and Y . Finally, since the point characterization used is based on the existence in $C(X - \{p\})$ of a function having no continuous extension at the point p , the conditions under which this occurs are analysed in new detail.

2. Point characterization in νX . The work of this section is motivated by observations made in the following examples.

EXAMPLE 2.1. Let X be the set of all positive, nonintegral, real numbers under the usual topology. Let Y be the same as X except that in addition to the integers also the number $1/2$ has been deleted. Then Y is homeomorphic to X and $C(Y)$ is isomorphic to $C(X)$, but νX , in addition to an imbedded image of Y , contains both a non-dense homeomorphic image of Y and a dense homeomorphic image of Y that is not an imbedded image of Y . Noting also that $p = 1/2$ is a G_δ point of X , one concludes that an isomorphism of $C(X - \{p\})$ with $C(X)$ does not imply that every continuous function on $X - \{p\}$ has a continuous extension at p .

EXAMPLE 2.2. Let Y be the open subinterval $(0, 1)$ of the real numbers under the usual topology. Form βY in the manner described in [4; 9] as a topologization of the collection of all dual ideals of open subsets of Y that are minimal in the collection of all finite, open, normal coverings of Y . Consider the function defined on Y by the formula $f(y) = y$. Let p be any point of βY such that $\tilde{f}(p) = 0$, where \tilde{f} denotes the continuous extension to all of βY of the function f . Finally, let p^* denote the point of βY which is the minimal dual ideal of open subsets of Y of the form $U^* = \{y \in Y \mid (1 - y) \in U\}$, where U is any open set in the ideal determining p . Clearly $p \neq p^*$.

Now let $X = \beta Y - \{p\}$. Then, as is easily surmised and will be confirmed in the next example, $\nu X = \beta Y$. From symmetry, it is clear that the homeomorphism mapping the point y of Y onto the point $1 - y$ of Y can be extended to a homeomorphism of βY onto βY which maps the point p onto the point p^* . It now follows that X has at least two distinct imbedded images in νX , one including p and excluding p^* , and a second including p^* and excluding p .

The next example is of a different type, but justifies, in what follows, a careful distinction between points p of X for which $C(X - \{p\})$ contains a bounded function lacking a continuous extension at p , and

a point p for which $C(X - \{p\})$ contains only unbounded functions lacking such extensions.

EXAMPLE 2.3. Let Y be a completely regular space for which $\beta Y \neq \nu Y$. Let p be a point of βY that is not in νY . Let X denote the subspace $Y + \{p\}$ of βY under the relative topology. Then every bounded function in $C(X - \{p\})$ has a continuous extension at p but at least one unbounded function in $C(X - \{p\})$ lacks such an extension. Moreover, the point p is a G_δ point in X . In fact, let p_0 be any point of a space X such that every bounded continuous function on $X - \{p_0\}$ has a continuous extension at p_0 , but such that some unbounded function f , continuous on $X - \{p_0\}$, lacks such an extension. Under these assumptions, it is easily seen that for every positive integer N either $f(p) > N$ throughout some deleted neighborhood of p_0 or that $f(p) < -N$ throughout such a neighborhood. Assume that for each N , $f(p) > N$ in such a neighborhood. Then with $1(p) \equiv 1$ on $X - \{p_0\}$ and forming $g = 1 \vee f$ in the usual manner, the function g is strictly positive on $X - \{p_0\}$. Finally, the function g^{-1} with $g^{-1}(p) = 1/g(p)$ for $p \neq p_0$, and with $g^{-1}(p_0) = 0$, is continuous on all of X and has p_0 as its unique zero point, thus establishing the above statement.

The diversity of homeomorphic images and of imbedded images in νX of even a single space Y with $C(Y)$ isomorphic to $C(X)$, as illustrated above, suggests that the imbedded subspaces of νX are best investigated on a pointwise basis, with special attention given to G_δ points. This is done in the following theorem.

THEOREM 1. (a) *The points of νX included in every imbedded subspace of νX are exactly the generalized G_δ points of νX .*

(b) *The G_δ points of νX are exactly the G_δ points of each imbedded subspace of νX .*

(c) *The generalized G_δ points of νX are exactly the generalized G_δ points of each imbedded subspace of νX .*

PROOF. (a) Clearly the isolated points of νX are found in every dense subspace of νX . Secondly, a point p of νX for which there exists a function defined and continuous on $\nu X - \{p\}$, but lacking a continuous extension at p , must obviously be included in every imbedded subspace. Finally, if the point p of νX is not a generalized G_δ point, then the deleted subspace $\nu X - \{p\}$ of νX is itself an imbedded subspace not including the point p .

(b) Let Y denote a subset of νX that is an imbedded subspace of νX . Then (a) states that every G_δ point of νX is a point of the subset Y , and it is clear that such a point remains a G_δ point in the topology of Y .

Conversely, let p_0 be a G_δ point of the imbedded subspace Y and let f be a continuous function on Y with $Z(f) = \{p_0\}$. Let \tilde{f} denote the continuous extension of f to vX , and let $(Z(f))^-$ denote the closure of $Z(f)$ in vX . Then, since it is known [3; 4] that $\tilde{f}(p) = 0$ at a point p of vX exactly when p is in $(Z(f))^- = \{p_0\}$, one concludes that p_0 as a point in vX is a G_δ point of vX .

(c) It is clear that every generalized G_δ point of vX is found in each imbedded subspace of vX and remains a generalized G_δ point in the relative topology of this subspace.

Conversely, let p_0 be a point in an imbedded subspace Y of vX that is a generalized G_δ point in the topology of Y . If p_0 is isolated in Y , it is clear that as a point of vX it is also isolated in vX . Hence assume that there is a function, defined and continuous on the deleted subspace $Y - \{p_0\}$, that does not have a continuous extension at p_0 .

Consider first the case where this function f is bounded. It must be shown that this function has a continuous extension to $vX - \{p_0\}$. This will be true if for each point p^* of $vX - Y$, regarded as a minimal dual ideal of open subsets of Y , there exists a number (directed limit) $\tilde{f}(p^*)$ such that for arbitrary positive ϵ there is an open set U in the dual ideal determining p^* such that $|\tilde{f}(p^*) - f(p)| < \epsilon$ for each point p of Y in U [4]. However, since vX is a Hausdorff space, there exist disjoint open sets U and U_0 of Y with $p_0 \in U_0$ while U is in the ideal determining p^* . Then there exists a function g continuous on Y with $g(p_0) = 0$ while $g(p) = 1$ for all $p \in U$. Then $f \cdot g$ is defined and continuous on $Y - \{p_0\}$ and, with $(f \cdot g)[p_0] = 0$, becomes continuous on Y with a continuous extension $(f \cdot g)^-$ to vX . Finally, the value $(f \cdot g)^-(p^*)$ of $(f \cdot g)^-$ at p^* clearly will serve as the desired value $\tilde{f}(p^*)$ of the directed limit indicated above.

Now suppose that the point p_0 of Y is such that every bounded function continuous on $Y - \{p_0\}$ has a continuous extension at p_0 , but that some unbounded function, continuous on $Y - \{p_0\}$, fails to have a continuous extension at p_0 . Then, as noted in Example 2.3, the point p_0 is a G_δ point of Y and as in vX is a G_δ point of vX .

3. Applications. In the studies of the isomorphisms of $C(X)$ and $C(Y)$ as determining a homeomorphism of vX and vY , frequent attention has been given to the question of when such an isomorphism determines a homeomorphism of X and Y , these latter spaces, of course, not being assumed to be Q -spaces [1; 6; 10]. The following theorem includes and extends the previous results on this question.

THEOREM 2. *If X and Y are completely regular, generalized G_δ spaces, then a ring, lattice or multiplicative semigroup isomorphism of $C(X)$ and $C(Y)$ determines a homeomorphism of X and Y .*

This theorem is an obvious consequence of Theorem 1. The fact that there are completely regular, generalized G_δ spaces that are not ordinary G_δ spaces is established by an example.

EXAMPLE 3.1. Let Y be the set of all points within an open circle of unit radius. With the center point excluded, regard each radius of this circle as an open subinterval of the real line and ascribe to the points on such a radius the corresponding topology. For the center point take as neighborhoods the entire interior of the circle less those points that lie on a finite number of arbitrarily designated radii at a distance from the center equalling or exceeding a positive number likewise arbitrarily designated for each such radius. It is easily verified that the set Y with this topology is a completely regular space in which each point, with the exception of the center point, is a G_δ point, while the center point is a generalized G_δ point but not a G_δ point.

As a second application of Theorem 1, for an arbitrary completely regular space X let μX denote the collection of all generalized G_δ points of X . Considered as a fixed subset of νX , this subset is the unique maximal subset of νX included in every imbedded subspace of νX . This subset, under its relative topology, will be an imbedded subspace of νX exactly when as a subspace of X it is dense in X and is such that every function continuous on this subspace has a continuous extension over all of X . When μX is an imbedded subspace of νX , it has a minimal property complementary to the maximal property of νX . This is understood in the sense that not only are $C(\mu X)$, $C(X)$ and $C(\nu X)$ isomorphic, but also that for any completely regular space Y the isomorphism of $C(Y)$ and $C(X)$ implies the inclusion relation $\mu X \subset \tilde{Y} \subset \nu X$ for every imbedded image \tilde{Y} of Y in νX .

For any generalized G_δ space X , one has $\mu X = X$, so that in this case μX has an imbedded image in νX and enjoys the described minimal property. However, for the space $X = \beta Y - \{p\}$ described in Example 2.2 it is easily verified that μX has no imbedded image in νX . In fact, in this example μX coincides with the initial space Y , and the space Y , as allowing unbounded continuous functions, cannot have an imbedded image in the space $\beta Y - \{p\}$, since the latter space allows only bounded continuous functions.³

4. Generalized G_δ points. The generalized G_δ points of a space X were defined in terms of the function space $C(X)$. In the following

³ The existence of such nonimbedded μX was first noted by M. Henriksen.

theorem, brief and incomplete note is made of characteristics of such points which distinguish them in terms of the topology of X .

THEOREM 3. (a) *A generalized G_δ point p of a space X such that every bounded continuous function on $X - \{p\}$ has a continuous extension at p , but such that some unbounded continuous function on $X - \{p\}$ lacks such an extension is, in fact, a G_δ point of X .*

(b) *A nonisolated point p of X has the property that every bounded continuous function on $X - \{p\}$ has a continuous extension at p if and only if every finite, open, normal covering of $X - \{p\}$ is obtained by deleting the point p from the sets in a finite, open, normal covering of X .*

Part (a) of Theorem 3 was established in Example 2.3. Here it is merely noted that, while a G_δ point p of the type described in (a) has the special properties that the first axiom of countability always fails at p and any continuous function f on X with $Z(f) = \{p\}$ has one-signed functional values in some neighborhood of p , nonetheless, examples may be constructed showing that neither of these properties characterize such G_δ points. What such a characterization would be is left here as an open question.

Part (b) of Theorem 3 is understood with relative ease when the space βX is regarded as a topologization of the set of all dual ideals of open sets of X that are minimal under the collection of all finite, open, normal coverings of X [4; 9]. Here it is merely noted that (b) is not equivalent to the statement: $\beta(X - \{p\}) = \beta X$. Thus in Example 2.1 with $p = 1/2$, there are many bounded continuous functions on $X - \{p\}$ lacking a continuous extension at p , while $\beta(X - \{p\})$ is clearly homeomorphic to βX .

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