ON THE BOUNDEDNESS OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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1. In this note we shall derive some criteria for the boundedness of solutions of nonlinear differential equations by means of a simple lemma given below. We shall also consider when a solution of a differential equation is unbounded.

We shall say that the function $h(x, r)$ possesses the property I if $h(x, r) \geqq 0$ for the specified range of values of x and r, if it is measurable in x for fixed $r \geqq 0$, continuous in r for fixed x, $x_0 \leqq x < \infty$, $r \geqq 0$, and if $r(x)$ is the maximal solution of the differential equation $r' = h(x, r)$ passing through the point $(x_0, 0)$.

Let us begin by proving the following lemma:

LEMMA. *Suppose that $h(x, r)$ has property* I. *Let $y(x)$ be continuous on $x_0 \leqq x < \infty$ and satisfy the inequality $|\Delta y(x)| \leqq \int_x^{x+\Delta x} h(t, y(t))dt$, $\Delta x \geqq 0$, then $y(x) \leqq r(x)$ for $x_0 \leqq x < \infty$.*

PROOF. The inequality shows that $y(x)$ is absolutely continuous in the interval $[x_0, \infty)$, which implies that the derivative $y'(x)$ exists almost everywhere in $[x_0, \infty)$. Furthermore, it is clear from the assumed inequality that the derivative satisfies the relation

$$|y'(x)| \leqq h(x, y(x)), \tag{1}$$

almost everywhere.

Suppose that $b(x, \epsilon)$ is a solution of $r' = h(x, r) + \epsilon$, $r(x_0) = 0$ where ϵ is an arbitrarily small quantity. It is easy to show that

$$y(x) \leqq b(x, \epsilon), \qquad x_0 \leqq x < \infty. \tag{2}$$

For suppose that this relation does not hold. Then, without loss of generality, let $[x_0, x_1]$ be an interval where $y(x) \geqq b(x, \epsilon)$. At x_0, we have $y(x_0) = b(x_0, \epsilon)$. Hence taking right-hand derivatives at x_0, we obtain the inequality

$$y'(x_0) \geqq b'(x_0, \epsilon). \tag{3}$$

From this we obtain the further inequality

$$h(x_0, y(x_0)) \geqq h(x_0, b(x_0, \epsilon)) + \epsilon, \tag{4}$$

which leads to a contradiction. Hence (2) holds.

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Since we know that $\lim_{\epsilon\to 0} b(x, \epsilon) = r(x)$, see [1], the lemma is proved.

Note. The notion of the maximal solution, and the above argument as used throughout this note, follow Kamke [1]. The lemma is a generalization of Bellman's lemma, cf. [2].

2. Let y and $f(x, y)$ be vectors with real components, $(y_1, y_2, \cdots, y_n)$ and $(f_1(x, y), f_2(x, y), \cdots, f_n(x, y))$ respectively. Define the norm of the vector y as follows: $\|y\| = \sum_{i=1}^{n} |y_i|$. Consider the system

$$y' = f(x, y), \qquad y(x_0) = 0, \tag{1}$$

where $f(x, y)$ is continuous on $x_0 \leqq x < \infty$, $\|y\| < \infty$.

Then we have

THEOREM 1. *Suppose that $h(x, r)$ has property* I *and that*

$$\|f(x, y)\| \leqq h(x, \|y\|). \tag{2}$$

Then if $r(x) = O(1)$ as $x \to \infty$, the norm of every solution of (1) *tends to a finite limit as $x \to \infty$. If, in particular, $r(x) = o(1)$ then each component of every solution of* (1) *tends to zero as $x \to \infty$.*

PROOF. Let a solution of (1) be $y(x) = \int_0^x f(t, y(t))dt$, and let $\Delta y(x) = y(x + \Delta x) - y(x)$, for $\Delta x > 0$. It follows that

$$\|\Delta y(x)\| \leqq \int_x^{x+\Delta x} \|f(t, y(t))\| dt \tag{3}$$

and hence that

$$\|\Delta y(x)\| \leqq \int_x^{x+\Delta} h(t, \|y(t)\|) dt. \tag{4}$$

Using the lemma derived above, we obtain

$$\|y(x)\| \leqq r(x), \qquad 0 \leqq x < \infty. \tag{5}$$

This together with the assumptions of the theorem yield the stated results, which generalize a result due to Wintner, [4].

The first part of the theorem is contained in [5] for the case where $h(x, r)$ is monotone nondecreasing in r. Here, we have merely assumed that $h(x, r)$ is continuous.

THEOREM 2. *Suppose that $h(x, r)$ has the property* I. *Let the differential system in* (1) *satisfy the further condition that*

$$\|f(x, y) - f(x, z)\| \leqq h(x, \|y - z\|). \tag{6}$$

Suppose that $r(x) = O(1)$ as $x \to \infty$. Then if one solution of (1) *tends to* *a finite limit as $x \to \infty$, then every solution vector tends to a finite limit. If, in particular, $r(x) = o(1)$ as $x \to \infty$, then every solution tends to the same finite limit as $x \to \infty$.*

PROOF. Let y and z be any two solutions of (1). Writing $v = y - z$ and proceeding as in the proof of the previous theorem, we obtain

$$\|\Delta v\| \leqq \int_x^{x+\Delta x} \|f(t, y) - f(t, z)\| dt \tag{7}$$

and, as a consequence,

$$\|\Delta v\| \leqq \int_x^{x+\Delta x} h(t, \|v\|) dt, \qquad \Delta x > 0. \tag{8}$$

Applying the lemma proved above, this yields

$$\|v\| = \|y - z\| \leqq r(x), \qquad x_0 \leqq x < \infty. \tag{9}$$

This together with the hypothesis of the theorem furnishes the desired result, which constitutes a generalization of a result of Wintner, [**3**]. When $n = 1$, the first part of the theorem is contained in [**5**].

COROLLARY. *If in the above theorem we suppose that $f(x, 0) \equiv 0$, then every solution tends to a finite limit as $x \to \infty$.*

Let us now consider when a solution of the differential equation can be unbounded as $x \to \infty$. We treat only the case $n = 1$.

THEOREM 3. *Let $h(x, r) > 0$ be continuous on $x_0 < x < \infty$, $r > 0$* and *$h(x, 0) \equiv 0$. Suppose that the following condition is satisfied:*

$$|f(x, y_2) - f(x, y_1)| \geqq h(x, |y_2 - y_1|). \tag{10}$$

Then, if any one solution of $r' = h(x, r)$ passing through $(x_0, 0)$ is unbounded as $x \to \infty$, then at least one solution of (1) is unbounded.

PROOF. Let y_1 and y_2 be any two solutions of (1). Put $z = y_2 - y_1$. Consider the equation

$$\begin{aligned} z' = y_2' - y_1' &= f(x, y_2) - f(x, y_1) \\ &= f(x, z + y_1) - f(x, y_1) = F(x, z). \end{aligned} \tag{11}$$

Using the assumption made above, we have $|F(x, z)| \geqq h(x, |z|)$. Since $F(x, z)$ is continuous and $h(x, r) > 0$, the above inequality shows that, for ordinates different from zero, $F(x, z) \gtrless 0$; which implies that either $F(x, z) \geqq h(x, |z|)$ or $F(x, z) \leqq -h(x, |z|)$. Let us consider the first case.

It may be shown by means of an argument similar to that given in the proof of the Lemma that $z(x, \epsilon) \geqq k(x)$ for $x_0 \leqq x < \infty$, where $k(x)$ is the given unbounded solution and $z(x, \epsilon)$ is a solution of $z' = F(x, z) + \epsilon$ passing through $(x_0, 0)$.

It follows that

$$z(x) = \lim_{\epsilon \to 0} z(x, \epsilon) \geqq k(x), \qquad x_0 \leqq x < \infty, \tag{12}$$

and that $z(x)$ is a solution of $z' = F(x, z)$ passing through $(x_0, 0)$. It follows that $z(x)$ is the required unbounded solution. The proof of the other case is similar.

3. In this section, we shall compare the solutions of two different systems. Let z and $g(x, z)$ be vectors with real components $(z_1, z_2, \cdots, z_n)$, $(g_1(x, z), g_2(x, z), \cdots, g_n(x, z))$, and let the norm be defined as above. Consider the system

$$z' = g(x, z),\ z(x_0) = 0, \tag{1}$$

where $g(x, z)$ is continuous on $x_0 \leqq x < \infty$, $|z| < \infty$.

THEOREM 4. *Suppose that $h(x, r)$ has property* I. *Let the functions $f(x, y)$ and $g(x, z)$ satisfy the condition*

$$\|f(x, y) - g(x, z)\| \leqq h(x, \|y - z\|), \tag{2}$$

and suppose that $r(x) = O(1)$ as $x \to \infty$.

Then, if one solution of (3.1) *tends to a finite limit as $x \to \infty$, then every solution of* (2.1) *tends to a finite limit as $x \to \infty$, and conversely. If, in particular, $r(x) = o(1)$ as $x \to \infty$, then every solution of* (2.1) *and* (3.1) *tends to the same finite limit as $x \to \infty$.*

PROOF. Suppose that y is a solution of (2.1) and that z is a solution of (3.1). Set $v = y - z$. Proceeding as above, it follows that

$$\|\Delta v\| \leqq \int_x^{x+\Delta x} h(t, \|v\|) dt, \qquad \Delta x > 0. \tag{3}$$

Our lemma yields

$$\|v\| \leqq r(x), \qquad x_0 \leqq x < \infty. \tag{4}$$

From this and our hypotheses, the desired conclusion follows.

COROLLARY. *If $g(x, 0) = 0$ and $r(x) = O(1)$ as $x \to \infty$, then every solution of* (2.1) *tends to a finite limit as $x \to \infty$.*

Note. A comparison theorem of this type is given in [**5**] for the case where $n = 1$ and $h(x, r)$ is monotonically nondecreasing in r.

Let us now consider another result concerning the unboundedness of a solution of a differential equation.

THEOREM 5. *Let* $h(x, r)>0$ *be continuous for* $x_0 \leqq x < \infty$, $r>0$, $h(x, 0)\equiv 0$, *and suppose that*

$$|f(x, y) - g(x, z)| \geqq h(x, |y - z|). \tag{5}$$

Let $g(x, 0)\equiv 0$. *Then the unboundedness of any one solution of* $r' = h(x, r)$ *through* $(x_0, 0)$, *apart from the minimal solution, guarantees the unboundedness of the maximal solution of* (2.1) *as* $x\to\infty$.

PROOF. Since $g(x, 0)\equiv 0$, we have by virtue of (5),

$$|f(x, y)| \geqq h(x, |y|). \tag{6}$$

By the argument similar to that of Theorem 3, this implies either $f(x, y)\geqq h(x, |y|)$ or $f(x, y)\leqq -h(x, |y|)$. Let us consider the first case. Proceeding as above, we obtain the inequality

$$y(x, \epsilon) \geqq k(x), \qquad x_0 \leqq x < \infty, \tag{7}$$

where $k(x)$ is the unbounded solution of $r' = h(x, r)$ and $y(x, \epsilon)$ is any solution of $y' = f(x, y)+\epsilon$ passing through $(x_0, 0)$. Since it is known that $\lim_{\epsilon\to 0} y(x, \epsilon) = z(x)$, the maximal solution of $z' = f(x, z)$ passing through $(x_0, 0)$, the conclusion follows from (7).

COROLLARY. *If the minimal solution of* $r' = h(x, r)$ *is unbounded, then every solution of* (2.1) *is unbounded as* $x\to\infty$.

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REFERENCES

1. E. Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig, 1930.
2. R. Bellman, *The boundedness of solutions of linear differential equations*, Duke Math. J. vol. 14 (1947).
3. A. Wintner, *Asymptotic equilibria*, Amer. J. Math. vol. 68 (1946).
4. ———, *An abelian lemma concerning asymptotic equilibria*, ibid.
5. B. Viswanatham, *On the asymptotic behaviour of solutions of non-linear differential equations*. Proceedings of the Indian Academy of Sciences vol. 36 (1952).

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