DUAL TRANSITIVITY IN FINITE PROJECTIVE PLANES

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Introduction. Let π be a finite projective plane of order n,—i.e., with $n+1$ points per line. The author [2] has shown that if π is doubly transitive and n is an odd nonsquare, then π is Desarguesian. M. Hall and D. Hughes, in a paper to appear soon, have removed the restriction that n must be odd. In this paper, we show that if π is dually transitive (see definition below), then it is doubly transitive.

1. Dual transitivity.

DEFINITION. Let p and p_1 be any two points, and let L and L_1 be any two lines such that p is not on L and p_1 is not on L_1. If (for all choices of p, p_1, L, L_1) there is a collineation of π which carries p into p_1 and L into L_1, then π will be said to be *dually transitive.*

THEOREM 1. *Let p and p_2 be any two points, and let L and L_2 be any two lines such that p is not on L and p_2 is not on L_2. If (for all choices of p, p_2, L, L_2) there is a correlation of π which carries p into L and p_2 into L_2 then π is dually transitive.*

PROOF. Consider the collineations which arise by taking the products of two correlations.

LEMMA 1. *Suppose that the projective plane π of order n admits a group Σ of collineations which* (1) *leaves a certain line L fixed and* (2) *is transitive on points not belonging to L, then if L_1 and L_2 are two lines $\neq L$ whose point of intersection lies on L, there is a collineation of Σ which carries L_1 into L_2.*

PROOF. Let us refer to points not on L as ordinary points. Let $\mathfrak{C}$ denote a transitive class of lines under Σ—i.e., each line of $\mathfrak{C}$ can be carried into each other line of $\mathfrak{C}$ by a collineation of Σ and no line not in $\mathfrak{C}$ is the image of any line in $\mathfrak{C}$. Now let p be an ordinary point which is on exactly k lines of $\mathfrak{C}$. The collineation which carries p into some other ordinary point p_1 carries the above mentioned k lines into k lines through p_1. Hence every ordinary point lies on exactly k lines of $\mathfrak{C}$. Let m denote the total number of lines in $\mathfrak{C}$. Since each line of $\mathfrak{C}$ contains n ordinary points, and there are n^2 ordinary points in all, it follows that

$$mn = kn^2, \quad \text{or} \quad m = kn.$$

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Now let L_1 be some line of $\mathfrak{C}$, intersecting L in the point q. Each of the n ordinary points of L_1 lies on $k-1$ lines of $\mathfrak{C}$ other than L_1 itself. This accounts for $n(k-1)$ lines of $\mathfrak{C}$. The remaining n lines must all go through q. Thus all of the ordinary lines which pass through any point q on L must belong to the same transitive class and the lemma is proved.

THEOREM 2. *If π is dually transitive, it is doubly transitive.*

PROOF. It follows from Lemma 1 that, given any line L and any point p on L, the group of collineations which leaves p and L fixed is transitive on the other lines through p. Now if p is on the lines L and L_1, by considering the product of the two groups which fix p and L on one hand, and p and L_1 on the other hand, we see that the group which fixes p is transitive on lines through p. The dual transitivity implies that the group Σ which leaves p fixed is transitive on lines not through p. Thus, there are exactly two transitive classes of lines under the group Σ. Parker [**3**] has shown that the number of point transitivity classes under a group of collineations is equal to the number of line classes. Since p is in a class by itself, all other points lie in a single transitive class under Σ. This in turn implies that any two points can be carried into any two points. (It should be noted here that D. R. Hughes has communicated a similar proof to the author.)

REFERENCES

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3. E. T. Parker, *On collineations of symmetric designs*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 250–351.

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