# ON THE HILBERT MATRIX, I

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1. For fixed $k<1$ the generalized Hilbert matrix is $H_k = ((m+n+1-k)^{-1})$, $m$, $n=0, 1, 2, \cdots$. By a *latent root* of $H_k$ we mean a complex number $\lambda$ such that there exists a non-null sequence of complex numbers $\{x_n\}_0^\infty$ with the property that

$$\sum_{n=0}^{\infty} (n+m+1-k)^{-1} x_n$$

converges to $\lambda x_m$ for all non-negative integers $m$. It is known (see [**6**; **3**], and [**4**]) that $\pi \csc \pi k$ is a latent root of $H_k$ if $k>0$. Taussky [**9**] posed the problem of determining whether $\pi$ is a latent root of $H_0$. This problem was solved by Kato [**5**], who applied a general theory to show that $H_k$ has the latent root $\pi$ when $1/2 \geqq k$.

We shall prove

THEOREM 1. *Every complex number with positive real part is a latent root of $H_k$.*

2. The Whittaker function $W_{k,m}$ is defined in [**11**, p. 340] by

$$\Gamma\left(m-k+\frac{1}{2}\right) W_{k,m}(x) x^{-m-1/2} = \int_{1/2}^{\infty} e^{-xs}\left(s+\frac{1}{2}\right)^{k+m-1/2}\left(s-\frac{1}{2}\right)^{m-k-1/2} ds, \tag{2.1}$$

where $k<1/2+\operatorname{Re} m$ and $\Gamma$ is the gamma function. For $n=0, 1, 2, \cdots$, let $\phi_n(x) = e^{-x/2}L_n(x)$, where $L_n$ is the $n$th Laguerre polynomial normalized so that the $L^2(0, \infty)$ inner product

$$(\phi_n, \phi_m) = \int_0^\infty e^{-t} L_n(t) L_m(t) dt = \delta_{n,m}.$$

If $x \geqq 0$

$$\int_0^\infty e^{-tx}\phi_n(t) dt = \left(x-\frac{1}{2}\right)^n\left(x+\frac{1}{2}\right)^{-n-1} \tag{2.2}$$

and $|\phi_n(x)| \leqq 1$ [**8**, p. 159].

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We define the operator $\mathcal{H}_k$ by

$$(2.3)\qquad (\mathcal{H}_k f)(x) = \Gamma(1-k)\int_0^\infty W_{k,1/2}(x+t)(x+t)^{-1}f(t)dt.$$

By 2.1, 2.2, and the Fubini theorem, if $x>0$, then

$$(2.4)\qquad \begin{aligned}(\mathcal{H}_k\phi_n)(x) &= \int_0^\infty\int_{1/2}^\infty\left(s+\frac{1}{2}\right)^k\left(s-\frac{1}{2}\right)^{-k}e^{-s(x+t)}ds\phi_n(t)dt\\ &= \int_{1/2}^\infty e^{-xs}\left(s-\frac{1}{2}\right)^{n-k}\left(s+\frac{1}{2}\right)^{k-n-1}ds\\ &= \Gamma(1-k+n)W_{k-n-1/2,0}(x)x^{-1/2},\end{aligned}$$

and by 2.4 and 2.2,

$$(2.5)\qquad \begin{aligned}(\mathcal{H}_k\phi_n,\phi_m) &= \int_0^\infty\int_{1/2}^\infty e^{-sx}\left(s-\frac{1}{2}\right)^{n-k}\left(s+\frac{1}{2}\right)^{k-n-1}ds\phi_n(x)dx\\ &= \int_{1/2}^\infty\left(s-\frac{1}{2}\right)^{n+m-k}\left(s+\frac{1}{2}\right)^{k-n-m-2}ds\\ &= (n+m+1-k)^{-1}.\end{aligned}$$

Thus if we consider $\mathcal{H}_k$ as an operator on $L^2(0,\infty)$, then $H_k$ is the matrix representation of $\mathcal{H}_k$ relative to the complete orthonormal set $\{\phi_n\}$. Henceforth we shall take $u$ to be a complex number such that $-1/2<\operatorname{Re} u<1/2$, $k<1$, and $f(x)=W_{k,u}(x)x^{-1}$. The equation

$$(2.6)\qquad \pi\sec\pi u f(x) = (\mathcal{H}_k f)(x)$$

is a particularization of an equation noted by Hari Shanker [**7**]. Hence a reasonable candidate for a solution $\{x_n\}$ of the matrix equation

$$(2.7)\qquad \sum_{n=0}^\infty (n+m+1-k)^{-1}x_n = \pi\sec\pi u x_n$$

is given by

$$(2.8)\qquad x_n = \int_0^\infty f(t)\phi_n(t)dt.$$

In the remainder of this note we shall show that indeed the $\{x_n\}$ defined by (2.8) satisfy (2.7).

3. From [**1**, Chapter 6], we know that $f(x)=O(x^{-1/2-|\operatorname{Re} u|})$ and $g(x)=W_{k-n-1/2,0}(x)x^{-1/2}=O(\log x)$ as $x\to 0$, and $f(x)=O(e^{-x/2}x^k)$,

$g(x)=O(e^{-x/2}x^k)$ as $x\to\infty$. It follows from these estimates that $f\in L(0,\infty)$ so

$$|x_n| \leqq \int_0^\infty |f(t)\phi_n(t)|\,dt \leqq \int_0^\infty |f(t)|\,dt < \infty,$$

and the $x_n$ are uniformly bounded. Also, the integrals in the following calculation are absolutely convergent so we may freely change the orders of integration. From (2.6), (2.8), and (2.4),

$$\begin{aligned}\pi \sec \pi u x_m &= \pi \sec \pi u \int_0^\infty f(x)\phi_m(x)dx\\ &= \int_0^\infty (\mathcal{H}_k f)(x)\phi_m(x)dx = \int_0^\infty f(x)(\mathcal{H}_k\phi_m)(x)dx \qquad (3.1)\\ &= \int_{1/2}^\infty\int_0^\infty e^{-sx}f(x)dx\left(s-\frac{1}{2}\right)^{m-k}\left(s+\frac{1}{2}\right)^{k-m-1}ds.\end{aligned}$$

Put $z=(s-1/2)(s+1/2)^{-1}$, so $s=2^{-1}(1+z)(1-z)^{-1}$ and

$$\pi \sec \pi u x_m$$
$$= \lim_{T\to 1-}\int_0^T\int_0^\infty \exp\left[-\frac{1}{2}x(1+z)(1-z)^{-1}\right]f(x)dx(1-z)^{-1}z^{m-k}dz.$$

But [8, p. 97]

$$\exp\left[-\frac{1}{2}x(1+z)(1-z)^{-1}\right](1-z)^{-1} = \sum_{n=0}^\infty z^n\phi_n(x),$$

where the series converges uniformly in $x$ and $z$ for $0\leqq x<\infty$, $0\leqq z\leqq T<1$. Hence

$$\begin{aligned}\pi \sec \pi u x_m &= \lim_{T\to 1-}\int_0^T\sum_{n=0}^\infty x_n z^{n+m-k}dz\\ &= \lim_{T\to 1-}\sum_{n=0}^\infty x_n\int_0^T z^{n+m-k}dz\\ &= \lim_{T\to 1-}\sum_{n=0}^\infty (n+m+1-k)^{-1}x_nT^{n+m+1-k}\\ &= \lim_{T\to 1-}\sum_{n=0}^\infty (n+m+1-k)^{-1}x_nT^n.\end{aligned}$$

Since the $x_n$ are uniformly bounded we may apply the Littlewood

Tauberian theorem [10, p. 233] to this last expression and infer that (2.7) is true. Finally, $w=\pi \sec \pi u$ maps the strip $-1/2<\operatorname{Re} u<1/2$ onto the open half-plane $0<\operatorname{Re} w$, so the proof of Theorem 1 is complete.

4. If we suppose $k-1/2<u<1/2$, $u\geqq 0$, then by (2.1) and (3.1), $f(x)$, $(x>0)$, and $x_0, x_1, x_2, \cdots$, are positive. Upon setting $\lambda=\pi \sec \pi u$ we have

THEOREM 2. *If $k<1/2$ and $\lambda\geqq\pi$, or if $1>k\geqq 1/2$ and $\lambda>\pi \csc \pi k$, then there exists a positive root vector $\{x_n\}$ corresponding to the latent root $\lambda$ of $H_k$.*

This theorem furnishes a solution to a problem posed by Kato in [5, p. 80].

5. I am indebted to the referee for

THEOREM 3. *Consider $H_k$ as a linear operator on the sequential Banach space $l^q$, where $2<q<\infty$. Then $H_k$ is bounded and $\pi \sec \pi u$ is an eigenvalue of $H_k$ whenever $|\operatorname{Re} u|<1/2-1/q$.*

PROOF. The boundedness of $H_k$ follows from [2, Theorem 364, p. 258]. The restriction on Re $u$ guarantees that $f\in L^p(0, \infty)$, where $p^{-1}+q^{-1}=1$. Since the $\phi_n$ are uniformly bounded it follows from F. Riesz's extension of the Hausdorff-Young theorem [12, p. 191] that $\{x_n\}$ given by (2.8) belongs to $l^q$. Finally, by 2.7, $\pi \sec \pi u$ is an eigenvalue of $H_k$.

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