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ON THE IDENTITY OF FUNCTION SPACES ON CARTESIAN PRODUCT SPACES[1]

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For $i=1, \cdots, n$, let S_i be a compact Hausdorff space and F_i be a closed linear subspace of the complex Banach space $C(S_i)$, the set of all continuous functions from S_i to the complex numbers. Let $S_1 \times \cdots \times S_n$ be the Cartesian Product of $S_1, \cdots, S_n$.

Define $F_1 * \cdots * F_n$ as $\{\phi \mid \phi \in C(S_1 \times \cdots \times S_n)$; for any $i=1, \cdots, n$ and $(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)$, $\phi(s_1, \cdots, s_{i-1}, \cdot, s_{i+1}, \cdots, s_n) \in F_i\}$. Also, define $F_1 \otimes \cdots \otimes F_n$ as the closure of the space of linear combinations of functions of the form $\phi(s_1, \cdots, s_n) = f_1(s_1) \times \cdots \times f_n(s_n)$ where each $f_i \in F_i$, where we base the topology on the norm, $\|\phi\| = \max_{s_1, \cdots, s_n} |\phi(s_1, \cdots, s_n)|$.

It is easily shown that $F_1 \otimes \cdots \otimes F_n$ is a subspace of $F_1 * \cdots * F_n$ and that if F_2 is one-dimensional, then $F_1 \otimes F_2 = F_1 * F_2$. Furthermore, by using continuous partitions of unity, one may show that $F_1 \otimes C(S_2) = F_1 * C(S_2)$. Therefore, if all but at most one of the F_i are either one-dimensional or $C(S_i)$, then $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$. However, it is not known whether for all cases $F_1 \otimes \cdots \otimes F_n$ will equal $F_1 * \cdots * F_n$ or not.[2] Although this question is not fully answered here, the purpose of this paper is to give a partial answer to this question. The results and arguments of this paper also apply to *real-valued* function spaces.

1. LEMMA. *Let F be a closed linear subspace of $C(S_1)$ and G a closed linear subspace of $C(S_2)$. Let H be a closed subspace of G differing from G by only one dimension. Then $F \otimes G = F * G$ implies that $F \otimes H = F * H$.*

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[2] Under Proposition 37 of his first paper of the Amer. Math. Soc. Memoirs, no. 16, Alexandre Grothendieck discusses a number of conjectures which are equivalent to this one.

PROOF. Choose a continuous linear functional x^* on G such that $\|x^*\|=1$ and $x^*(h)=0$ for $h\in H$. Choose $g_0\in G$ such that $x^*(g_0)=1$. Then every element of G will be of the form $h+\lambda g_0$ where $h\in H$ and λ is a complex number.

Choose $\phi\in F*H$ and $\epsilon>0$. Since $\phi\in F*G=F\otimes G$, choose $f_i\in F$, $g_i\in G$, $i=1,\cdots,k$ such that

$$\left\|\phi-\sum_{i=1}^{k} f_i\times g_i\right\|<\epsilon(1+\|g_0\|)^{-1}.$$

However, each g_i may be expressed as $h_i+\lambda_i g_0$ where $h_i\in H$. Therefore,

$$\sum_{i=1}^{k} f_i\times g_i=\sum_{i=1}^{k} f_i\times(h_i+\lambda_i g_0)=y+z\times g_0$$

where $y\in F\otimes H$ and $z\in F$.

For each $s\in S_1$,

$$\begin{aligned}|z(s)| &= |x^*[z(s)g_0]| = |x^*[y(s,\cdot)+z(s)g_0-\phi(s,\cdot)]| \\ &\leqq \|y+z\times g_0-\phi\|\,\|x^*\|<\epsilon(1+\|g_0\|)^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\sum_{i=1}^{k} f_i\times h_i-\phi\right\| &= \|y-\phi\| \\ &\leqq \|y+z\times g_0-\phi\|+\|z\times g_0\| = \|y+z\times g_0-\phi\|+\|z\|\|g_0\| \\ &< \epsilon(1+\|g_0\|)^{-1}+\epsilon\|g_0\|(1+\|g_0\|)^{-1}=\epsilon.\end{aligned}$$

Since $f_i\in F_i$ and $h_i\in H$, it follows that $\phi\in F\otimes H$. Therefore, $F\otimes H=F*H$.

2. LEMMA. *Let S_1, S_2, F, G and H be as in Lemma* 1. *Then $F\otimes H=F*H$ implies that $F\otimes G=F*G$.*

PROOF. Choose $g_0\in G-H$ and x^* as in the proof of Lemma 1. Since the set of continuous linear functionals on $C(S_2)$ is the set of regular measures on S_2, we may choose a measure μ such that $\int_{S_2} g(s_2)\mu(ds_2)=x^*(g)$ for each $g\in G$.

Consider an arbitrary element $\phi\in F*G$. For $s_1\in S_1$, define $\psi(s_1)$ as

$$\int_{S_2}\phi(s_1,s_2)\mu(ds_2).$$

Let μ' be any regular measure on S_1 such that for $f \in F$, $\int_{S_1} f(s_1)\mu'(ds_1) = 0$. Then using Fubini's Theorem, we get that

$$\int_{S_1} \psi(s_1)\mu'(ds_1) = \int_{S_1}\int_{S_2} \phi(s_1, s_2)\mu(ds_2)\mu'(ds_1)$$
$$= \int_{S_2}\int_{S_1} \phi(s_1, s_2)\mu'(ds_1)\mu(ds_2) = \int_{S_2} 0\mu(ds_2) = 0.$$

Therefore, $\psi \in F$.

For any $s_1 \in S_1$, it follows that

$$\int_{S_2} [\phi(s_1, s_2) - \psi(s_1)g_0(s_2)]\mu(ds_2) = \psi(s_1) - \psi(s_1) = 0$$

and so $\phi(s_1, \cdot) - \psi(s_1)g_0(\cdot)$ is an element of H. Therefore $\phi - \psi \times g_0 \in F * H = F \otimes H \subset F \otimes G$. Consequently, $\phi = [\phi - \psi \times g_0] + \psi \times g_0$ is an element of $F \otimes G$.

3. THEOREM. *For each $i = 1, \cdots, n$, let G_i and F_i be closed linear subspaces of $C(S_i)$ which differ from each other by at most a finite number of dimensions. Let no F_i or G_i be zero-dimensional.*

Then $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$ if and only if $G_1 \otimes \cdots \otimes G_n = G_1 * \cdots * G_n$.

PROOF. Let $H_1, \cdots, H_k$ be closed linear subspaces of $C(S_1)$ such that $H_1 = F_1$, $H_k = G_1$ and for $i = 1, \cdots, k-1$, H_i and H_{i+1} differ from each other by only one dimension. If $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$, then $F_2 \otimes \cdots \otimes F_n = F_2 * \cdots * F_n$. Using Lemma 1 or 2, we see that $H_2 \otimes F_2 \otimes \cdots \otimes F_n = H_2 * F_2 * \cdots * F_n$. Applying the lemmas in this way, a total of $k-1$ times, we see that $G_1 \otimes F_2 \otimes \cdots \otimes F_n = G_1 * F_2 * \cdots * F_n$. Now repeating this procedure $n-1$ more times, we obtain that $G_1 \otimes \cdots \otimes G_n = G_1 * \cdots * G_n$.

By repeating the above arguments, we see that $G_1 \otimes \cdots \otimes G_n = G_1 * \cdots * G_n$ implies $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$ and hence the theorem is proved.

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