THE INTERSECTION OF TWO CONVEX SURFACES AND PROPERTY P_3

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1. **Introduction.** The intersection of the surfaces of two convex bodies S_i $(i=1, 2)$ in an n-dimensional normed linear space L_n would appear to have a complicated structure. However, there exists a relatively simple approach to this matter via a concept which I have previously called "a three point convexity property P_3." See Definition 1 below or [6]. This concept applies to sets which include the set $S_1 \cup S_2$ as a special case. Hence, it not only yields simplicity but also adds generality. At first glance the result which is the simplest to state and the easiest to understand is the following.

THEOREM 0. *Let S_i $(i=1, 2)$ be two compact convex bodies in L_3 whose interiors have a nonempty intersection. Let B_i denote the boundary of S_i. If the intersection $B_1 \cdot B_2$ is contained in the interior of the convex hull of $S_1 \cup S_2$, then $B_1 \cdot B_2$ is the union of a finite number of disjoint simple closed curves.*

In the formal treatment of §3 we obtain the above theorem as a special case of a more general theorem for sets in L_n. In §4 a significant theorem about isolated points of local nonconvexity is obtained for closed sets having property P_3, and contained in a topological linear space.

2. **Definitions and résumé.** In order to achieve economy of effort, the following notations are used.

CONVENTIONS. Set theoretic product, union and difference are denoted by $\cdot$, $\cup$ and $\sim$ respectively. The letter L denotes a topological linear space, whereas L_n denotes a finite dimensional normed linear space of dimension n. A *variety* of L is a translate of a proper or improper linear subspace of L. *The interior of S relative to L is indicated by* int S, *whereas the interior of S relative to the minimal variety containing it is indicated by* intv S. Let $B(S)$ denote the boundary of S. The convex hull of S is indicated by conv S, the line segment determined by points x and y by xy, so that $xy = \{\lambda x + (1-\lambda)y,\ 0 \leqq \lambda \leqq 1\}$, and the corresponding line by $L(x, y)$. The segment xy is a crosscut of the complement of S if $x \in S$, $y \in S$ and if $S \cdot \text{intv}\ xy = 0$.

DEFINITION 1. *A set S in a linear space is said to possess property P_3*

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if for any three points x, y, z in S at least one of the segments xy, xz, yz is in S.

DEFINITION 2. *A set $S \subset L$ is locally convex at a point $p \in S$ if there exists a neighborhood N of p such that for each pair of points x and y in $S \cdot N$ it is true that $xy \subset S$. A point of S which is not a point of local convexity is called a point of local nonconvexity.*

Definition 2 is different from that used by Tietze [**5**] and others (see Klee [**4**]). This matter is discussed in §4, where it is significant.

DEFINITION 3. *The set of all points of local nonconvexity of S is denoted by Q.*

DEFINITION 4. *The convex kernel K of S is the set of all points z such that $zx \subset S$ whenever $x \in S$. (The set S is said to be starshaped with respect to each point of K. The convex kernel of a set $S \subset L$ is convex. See Brunn* [**3**].)

RÉSUMÉ. In my previous paper [**6**], the study was primarily restricted to sets in E_2, the Euclidean plane. The treatment there was relatively complete as far as E_2. However, several results proved there also hold in L. We need them in §3 where $S \subset L_n$, and in §4 where $S \subset L$.

A. *Let S be a closed set in a topological linear space L. If S has property P_3, then $Q \subset K$. Furthermore, we have $Q \subset B(S)$, so that $Q \subset B(K)$.*

B. *If xy is a crosscut of the complement of the set S in statement* A, *and if $z \in K$, then the triangle* $\operatorname{conv}(x \cup y \cup z)$ *contains a point $q \in Q$ such that if $a \in \operatorname{intv} xq$, $b \in \operatorname{intv} yq$, then $ab \not\subset S$.*

In Remark A, the fact $Q \subset K$ follows readily from Definitions 1–4. The fact $Q \subset B(S)$ is trivial if $S \subset L_n$; if $S \subset L$, then it follows quickly from elementary properties of the neighborhoods of L (see §4 and the proof of Theorem 2; also [**2**] and [**7**]). The proof of Remark B involves only a two-dimensional variety of L, and hence it will not be repeated.

3. **Main theorem.** The following sequence of lemmas gives us an elementary breakdown of the theory leading to Theorem 1.

LEMMA 1. *Suppose $S \subset L_n$ is a closed set having property P_3. Let Δ be an r-dimensional simplex whose vertices $x_1, \cdots, x_{r+1}$ belong to S, and suppose $t \in \operatorname{intv} \Delta$. Then there exist points $u_i \in S \cdot B(\Delta)$, $(i=1, 2)$, such that $t \in \operatorname{intv} u_1u_2$.*

PROOF. The result is clearly true for $r=1$. So suppose it is true for simplices up to and including those of dimension $r-1$. Let $L(x_1, t) \cdot \operatorname{conv}(\bigcup_2^{r+1} x_\alpha) \equiv s$. Since $t \in \operatorname{intv} \Delta$, the set s is a point, and $s \in \operatorname{intv} \operatorname{conv}(\bigcup_2^{r+1} x_\alpha)$. By our induction assumption there exist points v_i ($i=1, 2$) such that $v_i \in S \cdot B(\text{conv } (\bigcup_2^{r+1} x_\alpha))$, and such that $s \in \text{intv } v_1v_2$. If $v_1v_2 \subset S$ then $s \in S$, and if we let $x_1 = u_1$, $s = u_2$, we have $t \in \text{intv } u_1u_2$ with $u_i \in B(\Delta) \cdot S$. If, however, $s \notin S$, then by property P_3 either x_1v_1 or x_1v_2 belongs to S. Without loss, suppose $x_1v_1 \subset S$. The set $L(v_2, t) \cdot x_1v_1 \equiv u_1$ is a point since $t \in \text{intv conv } (x_1 \cup v_1 \cup v_2)$. Letting $v_2 \equiv u_2$, we get $t \in \text{intv } u_1u_2$, and it is easy to prove that $u_i \in B(\Delta) \cdot S$.

Standing hypothesis in Lemmas 2–5. We assume in Lemmas 2–5 that S is a compact set in L_n, and that S has property P_3.

LEMMA 2. *Suppose* int $K \neq 0$ *where K is the convex kernel of S. If* $q \in Q \cdot \text{int conv } S$, *then there exist points* $x \in S$, $y \in S$, $z \in \text{int } K$ *such that xy is a crosscut of the complement of S, and such that*

$$q \in \text{intv conv } (x \cup y \cup z).$$

PROOF. Choose a point $z \in \text{int } K$, and let $R(z, q)$ be the ray containing zq and having z as its endpoint. Let $R(z, q) \cdot B(\text{conv } S) \equiv t$. Since $zq \subset \text{int conv } S$, it follows that t is a point and $t \in B(\text{conv } S)$. Since S is compact, the conv S is compact; hence $t \in \text{conv } S$. Since intv $zq \subset \text{int } K$, and since $q \in B(S)$, it follows that $t \notin S$. Let $H(t)$ be a hyperplane of support to conv S at t. Since $t \in H(t) \cdot \text{conv } S$, let Δ be an r-dimensional simplex ($r \leqq n-1$) of minimal dimension in $H(t)$ whose vertices belong to $H(t) \cdot S$, and which contains t (Δ must exist; see Bonnesen and Fenchel [1, p. 9]). Since $t \notin S$, and since Δ has minimal dimension, we have $t \in \text{intv } \Delta$. Moreover, $H(t) \cdot S$ is a closed set satisfying property P_3. Hence, by Lemma 1, there exist points $u_i \in B(\Delta) \cdot S$ such that $t \in \text{intv } u_1u_2$. Since $t \notin S$, and since S is closed, there exists a crosscut xy ($xy \subset u_1u_2$) of the complement of S such that $t \in \text{intv } xy$. Furthermore, since $z \notin H(t)$, and since $q \in \text{intv } zt$, we have $q \in \text{intv conv } (x \cup y \cup z)$. This completes the proof.

LEMMA 3. *If xy is a crosscut of the complement of the set S, and if* $z \in \text{int } K$, *then* $Q \cdot \text{conv } (x \cup y \cup z) \equiv q$ *is a point, and*

$$q \in \text{intv conv } (x \cup y \cup z).$$

PROOF. Since $z \in \text{int } K$, let $N(z)$ be a spherical neighborhood of z such that $N(z) \subset K$. Remark B of §2 states that $Q \cdot \text{conv } (x \cup y \cup z) \neq 0$. Suppose two points $q_i \in Q \cdot \text{conv } (x \cup y \cup z)$ ($i=1, 2$) exist. Since int conv $(x \cup N(z)) \subset \text{int } S$, and since int conv $(y \cup N(z)) \subset \text{int } S$, we must have $q_i \cdot \text{intv } xz = 0$, $q_i \cdot \text{intv } yz = 0$, otherwise $q_i \notin B(S)$. Moreover, $q_i \neq z$. Also since $xy \not\subset S$, and since $q_i \in K$, we must have $q_i \neq x$, $q_i \neq y$. Hence $q_i \in \text{intv conv } (x \cup y \cup z)$. This implies that at least one of the following conditions holds, $L(q_1, q_2) \cdot \text{intv } xz \neq 0$, $L(q_1, q_2) \cdot \text{intv } yz \neq 0$, $L(q_1, q_2) \cdot z \neq 0$. Suppose, without loss of generality, that 0

$\neq L(q_1, q_2)\cdot(z\cup \text{intv } xz)\equiv s$, and that $q_2 \in \text{intv } q_1 s$. Let $N(s)$ be a spherical neighborhood of s such that $N(s)\subset \text{conv } (x\cup N(z))\subset S$. However, this implies that $q_2 \in \text{int conv } (q_1\cup N(s))\subset \text{int } S$, which contradicts the fact $q_2\in B(S)$. Hence, we have $Q\cdot \text{conv } (x\cup y\cup z)\equiv q$, a single point, such that $q\in \text{intv conv } (x\cup y\cup z)$.

LEMMA 4. *Suppose* int $K\neq 0$ *for the set* S, *and suppose*

$$q \in Q\cdot \text{int conv } S.$$

Then there exists a closed convex body $M(q)$ *such that* $q\in \text{int } M(q)$ *and such that* $Q\cdot M(q)$ *is homeomorphic to an* $(n-2)$*-dimensional closed spherical cell whose center corresponds to* q.

PROOF. Consider the three points x, y, z of Lemma 2, and let $N(z)$ be a closed n-sphere with center at z such that $N(z)\subset \text{int } K$, and such that $N(z)\cdot \text{conv } (x \cup y \cup q) = 0$, where, by Lemma 3, $q \in Q \cdot \text{intv conv } (x\cup y\cup z)$. The two dimensional variety determined by x, y, z is denoted by $L_2(x, y, z)$. Let L_{n-2} be an $(n-2)$-dimensional variety such that $L_2(x, y, z)\cdot L_{n-2}=z$. The set $L_{n-2}\cdot N(z)\equiv C_{n-2}(z)$ is an $(n-2)$-dimensional sphere with center z. The set $\text{conv}(xy\cup C_{n-2}(z)) \equiv M(q)$ is an n-dimensional convex body containing q in its interior. We will prove that $M(q)$ has the property stated in the lemma. First, for any point $p\in C_{n-2}(z)$, Lemma 3 implies that there exists a unique point $f(p)\in Q\cdot M(q)$ such that $f(p)\in \text{intv conv}(x\cup y\cup p)$. Furthermore for each point $q'\in Q\cdot M(q)$ there exists a unique point $p\in C_{n-2}(z)$ such that $q'\in \text{intv conv}(x\cup y\cup p)$, because $L_2(x, y, q')\cdot C_{n-2}(z) \equiv p\in C_{n-2}(z)$ and because $M(q)=\{\lambda A+(1-\lambda)B,\ 0\leqq\lambda\leqq 1,\ A=xy,\ B=C_{n-2}(z)\}$. Hence, the function $\{f(p): p\in C_{n-2}(z)\}$ is a biunique mapping of $C_{n-2}(z)$ onto $Q\cdot M(q)$. To prove it is also continuous for $p\in C_{n-2}(z)$, let $p_i\in C_{n-2}(z)$ be a sequence such that $p_i\to p$ as $i\to\infty$. Since $L_2(x, y, p_i)\to L_2(x, y, p)$ as $i\to\infty$, the compactness of $Q\cdot M(q)$ implies that $f(p_i)\to f(p)$, otherwise $\text{conv}(x\cup y\cup p)$ would contain at least two distinct points of Q, in violation of Lemma 3. The inverse of $f(p)$ is thus also continuous since L_n is a metric space. This completes the proof.

LEMMA 5. *For any* $\epsilon>0$, *the set* $M(q)$ *in Lemma* 4 *can be chosen so that the diameter of* $M(q)$ *is less than* ϵ.

PROOF. In the proof of Lemma 4, we may choose $z'\in \text{intv } qz\subset \text{int } K$, and $C_n(z')\subset \text{int } K$, $x'y'\subset \text{conv}(x\cup y\cup q)$ so that

$$M'(q) \equiv \text{conv } (x'y' \cup C_{n-2}(z'))$$

has diameter less than ϵ, and so that $Q\cdot M'(q)$ is homeomorphic to $C_{n-2}(z')$.

DEFINITION 5. *A set* $\mathfrak{M} \subset L_n$ *is said to be a closed* $(n-2)$*-dimensional manifold, if* $\mathfrak{M}$ *is a compact connected set, and if for each* $\epsilon > 0$ *each point* $x \in \mathfrak{M}$ *is interior to an n-dimensional closed convex set* $M(x)$ *of diameter less than* ϵ, *such that* $\mathfrak{M} \cdot M(x)$ *is homeomorphic to a closed* $(n-2)$*-dimensional spherical cell in* L_{n-2} *whose center corresponds to* x.

THEOREM 1. H_1. *Suppose* S *is a compact set in a finite dimensional normed linear space* L_n, *and suppose* S *has property* P_3.

H_2. *Assume that the convex kernel of* S *has interior points* (*i.e.* int $K \neq 0$).

H_3. *Finally, assume that all the points of local nonconvexity of* S *are interior to the convex hull of* S (*i.e.* $Q \subset$ int conv S).

C. *Then the set of points of local nonconvexity of* S *consists of a finite number of disjoint closed* $(n-2)$*-dimensional manifolds.*

PROOF. First, the compactness of S and the definition of local nonconvexity imply immediately that the set Q is compact. Let Q_1 denote a component of Q. Lemma 5 implies that Q_1 is a closed $(n-2)$-dimensional manifold in L_n. Since Q is compact, it can be covered by a finite subset of the neighborhoods defined in Lemma 4. This implies that Q will have a finite number of components.

COROLLARY 1. *Let* S_1 *and* S_2 *be two compact convex bodies in* L_n *whose interiors have a nonempty intersection. If* $B(S_1) \cdot B(S_2)$ *is contained in the interior of the convex hull of* $S_1 \cup S_2$, *then* $B(S_1) \cdot B(S_2)$ *is the union of a finite number of disjoint closed* $(n-2)$*-dimensional manifolds.*

PROOF. The set $S_1 \cup S_2$ satisfies hypotheses H_i $(i=1, 2, 3)$ of Theorem 1.

Theorem 0 in the introduction can be obtained from Theorem 1 as follows. For sets in L_3 Theorem 1 implies readily that each component Q_1 of Q must be homeomorphic to the boundary of a circular two-cell. Since K is a compact convex body (int $K \neq 0$) in L_3, and since $Q_1 \subset B(K)$, the Jordan curve theorem as applied to the two-dimensional surface $B(K)$ implies that Q_1 is a simple closed curve in L_3 (no knots are possible).

4. **Isolated points of local nonconvexity.** In order to describe in a topological linear space L what the existence of an *isolated* point of local nonconvexity does to the totality of points of local nonconvexity of S, the following definitions will be needed.

DEFINITION 6. *A space* L *is locally starlike if for each neighborhood* $U(p)$ *of* p *there exists a neighborhood* $V(p)$, *starshaped from* p, *such that* $p \in V(p) \subset U(p)$. *A corresponding definition holds for a locally convex* *space L*. The latter concept is not to be confused with that of Definition 2, and the usage below will clearly avoid any such confusion.

DEFINITION 7. *The set $S \subset L$ is strongly locally convex at a point $p \in S$ if there exists a neighborhood N of p such that $S \cdot N$ is convex. A point $p \in S$ which is not a point of strong local convexity is called a point of mild local nonconvexity.*

In a topological linear space L which is locally convex in the sense of Definition 6, Definitions 2 and 7 are equivalent. However, in general they are not equivalent. For instance, each neighborhood in a nonlocally convex topological linear space (in the sense of Definition 6) is locally convex in the sense of Definition 2 but not strongly locally convex in the sense of Definition 7. The following theorem establishes a significant difference between sets S (having property P_3) in L_2 and in L, where dimension $L > 2$.

THEOREM 2. *Let S be a closed set in a topological linear space L, where dimension $L > 2$. Assume that S has property P_3, and that S is not contained in any two-dimensional variety of L.*

If the set Q of points of local nonconvexity of S has an isolated point, then Q has at most two points (*see Definitions* 2 *and* 3).

PROOF. Let p be an isolated point of Q. Since a topological linear space is locally starlike [**2**; **7**], let $N(p)$ be a neighborhood of p, starshaped from p, such that $N(p) \cdot Q = p$. Since L is a topological linear space, it is well known [**2**; **7**] that there exists a neighborhood $N_1(p) \subset N(p)$, starshaped from p, such that for any two points $u \in N_1(p)$, $v \in N_1(p)$ we have $uv \subset N(p)$. Since p is a point of local nonconvexity of S, there exist points x_1, y_1 in $S \cdot N_1(p)$ such that $x_1y_1 \not\subset S$, and moreover from the preceding sentence we have $x_1y_1 \subset N(p)$. Since $x_1y_1 \not\subset S$, there exists a segment $xy \subset x_1y_1$ such that $x \in S$, $y \in S$, $S \cdot \operatorname{intv} xy = 0$, $xy \subset N(p)$, even though xy may not belong to $N_1(p)$. Since $N(p)$ is starshaped from p, and since $xy \subset N(p)$, we have $\operatorname{conv}(x \cup y \cup p) \subset N(p)$. Since $p \notin L(x, y)$, let $L(x, y, p)$ denote the variety determined by x, y, p. We next prove that $Q \subset L(x, y, p)$. Suppose $Q \not\subset L(x, y, p)$, and choose $q \in Q \sim L(x, y, p)$. Let $L(x, y, p, q)$ denote the three-dimensional variety determined by x, y, p, q. Since $L(x, y, p, q)$ is locally convex relative to the topology obtained by intersecting neighborhoods of L by $L(x, y, p, q)$, there exists, relative to $L(x, y, p, q)$, a convex neighborhood $V(p) \subset N(p) \cdot L(x, y, p, q)$, $p \in V(p)$. Since $N(p) \cdot Q = p$, and since $\operatorname{conv}(x \cup y \cup p) \subset N(p)$, we have $Q \cdot \operatorname{conv}(x \cup y \cup p) = p$. Hence, by Remark B of §2 there exist points $a \in V(p) \cdot \operatorname{intv} xp$, $b \in V(p) \cdot \operatorname{intv} yp$ such that $ab \not\subset S$. Hence, a crosscut cd of the complement of S exists in $V(p) \cdot \operatorname{conv}(x \cup y \cup p)$. Choose a point $r \in V(p) \cdot \operatorname{intv} pq$. Since $pq \subset K$, and since $c \in S$, $d \in S$, $S \cdot \operatorname{intv} cd = 0$, Remark B of §2 implies that $Q \cdot \operatorname{conv}(c \cup d \cup r) \neq 0$. Let $q_1 \in Q \cdot \operatorname{conv}(c \cup d \cup r)$. Since $V(p)$ is convex, we have $q_1 \in V(p) \subset N(p)$ with $q_1 \neq p$. This contradicts the fact $Q \cdot N(p) = p$. Hence, we have shown that $Q \subset L(x, y, p)$.

Secondly, to prove that Q is contained in a line, choose a point $s \in S \sim L(x, y, p)$ since $S \not\subset L(x, y, p)$. As in the preceding paragraph, let $V(p)$ be a three-dimensional convex neighborhood of p such that $V(p) \subset N(p) \cdot L(x, y, p, s)$. Choose $z \in V(p) \cdot \operatorname{intv} sp$. By Remark B, just as in the preceding paragraph, there exist points $a \in V(p) \cdot \operatorname{intv} xp$, $b \in V(p) \cdot \operatorname{intv} yp$ such that $ab \not\subset S$. Since $ab \not\subset S$, property P_3 implies that we may, without loss, assume $az \subset S$. Since S is closed, and since $ab \not\subset S$, there exists a point $h \in \operatorname{intv} az$ such that $bh \not\subset S$. Since $\operatorname{conv}(a \cup b \cup z) \subset V(p) \subset N(p)$, we have $bh \subset N(p)$. Hence, since $bh \not\subset S$, the reasoning that implies $Q \subset L(x, y, p)$ also implies $Q \subset L(b, h, p)$. Since $L(x, y, p) \cdot L(b, h, p) = L(y, p)$, we have $Q \subset L(y, p)$.

Finally, to show that Q consists of at most two points, suppose points $q \in Q$, $r \in Q$ exist with r between p and q on $L(p, q)$. Since $Q \subset L(y, p)$, we have $L(p, q) = L(y, p)$. Since $K \cdot \operatorname{intv} yp = 0$, we have $p \in \operatorname{intv} yr$. The set $L(p, q) \sim \operatorname{intv} pq$ is closed; hence, let $N(r)$ be a neighborhood of r, starshaped from r, such that $N(r) \cdot (L(p, q) \sim \operatorname{intv} pq) = 0$. There exists a neighborhood $N_1(r)$ of r such that $\lambda N_1(r) + (1-\lambda) N_1(r) \subset N(r)$ $(0 \leqq \lambda \leqq 1)$ [7]. Since $r \in Q$, there exist points $u \in S \cdot N_1(r)$, $v \in S \cdot N_1(r)$ such that $uv \not\subset S$. However, $uv \subset N(r)$, so that $uv \cdot (L(p, q) \sim \operatorname{intv} pq) = 0$. Moreover, since $pq \subset K$, and since $uv \not\subset S$, we have $uv \cdot pq = 0$, so that $uv \cdot L(p, q) = 0$. Hence, since $Q \subset L(y, p)$, it is true that $Q \cdot \operatorname{conv}(u \cup v \cup p) = p$. Hence, the local convexity (in the sense of Definition 6) of the variety $L(u, v, p)$ in the two-dimensional relative topology and Remark B imply the existence of points $u' \cup v' \subset S \cdot N(p)$ such that $u' \cup v' \subset \operatorname{conv}(u \cup v \cup p)$ and $S \cdot \operatorname{intv} u'v' = 0$, $u'v' \subset N(p)$. Hence, the exact sequence of reasons that implies $Q \subset L(y, p) \subset L(x, y, p)$ also implies that $Q \subset L(u', v', p)$ and that either $Q \subset L(u', p)$ or $Q \subset L(v', p)$. Suppose that $Q \subset L(u', p)$. Then since $u' \notin L(y, p)$, we have $Q \subset L(y, p) \cdot L(u', p) = p$. This is a contradiction since $q \in Q$. Hence, Q has at most two points.

It should be observed that Theorem 2 does not hold for closed sets $S \subset L_2$ having Property P_3. A closed two-cell whose boundary is the conventional m-pointed star has property P_3, yet it has m isolated points of local nonconvexity [6].

Theorem 2 and the remarks following Definition 7 imply the following result.

THEOREM 3. *Let S be a closed set in a locally convex* (*in the sense of Definition* 6) *topological linear space L, where dimension $L>2$. Assume that S has property P_3, and that S is not contained in any two-dimensional variety of L.*

If the set of points of mild local nonconvexity of S has an isolated point, then S has at most two points of mild local nonconvexity.

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