A NOTE ON THE GAUSS-GREEN THEOREM[1]

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A very general measuretheoretic version of the Gauss-Green formula (for n-space) has recently been studied by De Giorgi in [**DG 1**] and [**DG 2**]. Independently Fleming and Young have obtained related results (for 3-space) in [**FY**], employing the technique of "generalized surfaces." Here the work of De Giorgi will be supplemented through use of the geometric concept of exterior normal introduced in [**F 1**].

Assuming that A is a subset of Euclidean n-space E_n, measurable with respect to Lebesgue measure L_n, consider the following two conditions:

(1) *There exist finite real valued (signed) Borel measures* $\Phi_1, \cdots, \Phi_n$ *over* E_n *such that*

$$\int_A D_i f dL_n = \int_{E_n} f d\Phi_i \qquad \textit{for } i = 1, \cdots, n$$

whenever f is a continuously differentiable function on E_n which vanishes at infinity. [In the Schwartz language of distributions this means that the partial derivatives of the characteristic function of A are the measures $-\Phi_1, \cdots, -\Phi_n$.]

(2) *There exists a number M and an infinite sequence of sets A_j with polyhderal (or smooth) boundaries B_j such that*

$$L_n[(A - A_j) \cup (A_j - A)] \to 0 \textit{ as } j \to \infty$$

and

$$H_n^{n-1}(B_j) \leqq M \textit{ for all positive integers } j,$$

where H_n^{n-1} is $n-1$ dimensional Hausdorff measure over E_n.

De Giorgi proved in [**DG 1**] that the conditions (1) and (2) are equivalent. He also showed that the total variation over E_n of the vectorvalued measure Φ defined by $\Phi_1, \cdots, \Phi_n$ in (1) is equal to the infimum of all numbers M suitable for (2). In his subsequent paper [**DG 2**], he established various interesting local properties of Φ, which will be used and extended presently.

Say, as in [**F 1**], that a unit vector u is an *exterior normal of A at x* if and only if

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$$r^{-n}L_n(\{y \mid |y-x| < r,\ (y-x)\cdot u < 0,\ y \notin A\}) \to 0$$

and

$$r^{-n}L_n(\{y \mid |y-x| < r,\ (y-x)\cdot u > 0,\ y \in A\}) \to 0$$

as $r \to 0+$, where $\cdot$ is the inner product. Such a unit vector u, if it exists, is uniquely determined by A and x, and denoted $\nu(A, x)$. In case no such u exists, $\nu(A, x)$ is the null vector. This defines for each $x \in E_n$ a vector $\nu(A, x)$ with components $\nu_1(A, x), \cdots, \nu_n(A, x)$.

It will be shown that (1) and (2) are equivalent to the condition:

$$H_n^{n-1}(\{x \mid |\nu(A, x)| = 1\}) < \infty \tag{3}$$

and

$$\int_A D_i f dL_n = \int_{E_n} f(x)\nu_i(A, x) dH_n^{n-1}x \qquad \text{for } i = 1, \cdots, n,$$

whenever f is a continuously differentiable function on E_n which vanishes at infinity.

Obviously (3) implies (1) with

$$\Phi_i(S) = \int_S \nu_i(A, x) dH_n^{n-1}x$$

whenever S is a Borel set contained in E_n.

The fact that (1) implies (3) will be established with the help of the following lemma:

If (1) *holds and if* $|\nu(A, x)| = 1$, *then*

$$\limsup_{r\to 0^+} r^{-n+1} | \Phi[K(x, r)] | \geqq \frac{\alpha(n)}{4},$$

where $K(x, r) = \{y \mid |y-x| < r\}$ *and* $\alpha(n) = L_n[K(x, 1)]$.

In the proof of this lemma one may assume, to simplify the notation, that x is the origin and $\nu(A, x)$ is the nth base vector $(0, \cdots, 0, 1)$. According to [**DG 2**, Lemma III] it is true for L_1 almost all positive numbers r that

$$\int_{A\cap K(x,r)} D_n f\, dL_n = \int_{K(x,r)} f\, d\Phi_n + \int_A f(y)\nu_n[K(x, r), y] dH_n^{n-1}y$$

whenever f is a continuously differentiable function on E_n which vanishes at infinity. Moreover, since $A \cap K(x, r)$ is bounded, vanishing at infinity is irrelevant; and continuously differentiable may be generalized to Lipschitzian, by smoothing. In particular the formula applies with

$$f(y) = y_n + \left| r^2 - \sum_{i=1}^{n-1} (y_i)^2 \right|^{1/2} \text{ for } y \in E_n.$$

Consider the hemispherical shells

$$V(r) = \{y \mid |y| = r \text{ and } y_n \leqq 0\},$$
$$W(r) = \{y \mid |y| = r \text{ and } y_n > 0\},$$

and note that

$$f(y) = 0 \text{ for } y \in V(r), \qquad |f(y)| \leqq 2r \text{ for } y \in K(x, r),$$

$$L_n[A \cap \{y \mid |y| < s \text{ and } y_n > 0\}] = \int_0^s H_n^{n-1}[A \cap W(r)]dr \text{ for } s > 0.$$

Now let $\epsilon > 0$. Inasmuch as

$$L_n[A \cap \{y \mid |y| < s \text{ and } y_n > 0\}] \leqq \frac{\epsilon}{n} s^n = \int_0^s \epsilon r^{n-1} dr$$

for small $s > 0$, there exist arbitrarily small $r > 0$ for which

$$H_n^{n-1}[A \cap W(r)] \leqq \epsilon r^{n-1},$$

which are unexceptional with respect to [**DG 2**, Lemma III], and for which

$$L_n[A \cap K(x, r)] \geqq \frac{r^n}{2} [\alpha(n) - \epsilon].$$

For such r it follows that

$$\int_{A \cap K(x,r)} D_n f \, dL_n = L_n[A \cap K(x, r)] \geqq \frac{r^n}{2} [\alpha(n) - \epsilon],$$

$$\int_{K(x,r)} f \, d\Phi_n \leqq 2r |\Phi_n[K(x, r)]| \leqq 2r |\Phi[K(x, r)]|,$$

$$\int_A f(y) \nu_n[K(x, r), y] dH_n^{n-1} y \leqq 2r H_n^{n-1}[A \cap W(r)] \leqq 2\epsilon r^n,$$

hence

$$r^{-n+1} |\Phi[K(x, r)]| \geqq \frac{1}{4} [\alpha(n) - \epsilon] - \epsilon = \frac{1}{4} [\alpha(n) - 5\epsilon].$$

Now suppose (1) holds and let μ be the total variation measure associated with the vector valued measure Φ. The preceding lemma implies that the $n-1$ dimensional μ density of E_n is no less than

$\alpha(n)/[4\alpha(n-1)]$ at each point of the set

$$N = \{x \mid |\nu(A, x)| = 1\}.$$

If S is any Borel set contained in N, then $\mu(E_n - S) < \infty$, and it follows from [**F 3**, 3.3 and 3.1], that

$$H_n^{n-1}(S) \leqq \frac{\alpha(n)}{4\alpha(n-1)} \mu(S).$$

Furthermore De Giorgi established in [**DG 2**, Theorems III and IV] the existence of a Borel set F such that

$$\mu(E_n - F) = 0, \qquad F \subset N$$

and

$$\mu(S) = H_n^{n-1}(S), \qquad \Phi_i(S) = \int_S \nu_i(A, x) dH_n^{n-1} x \quad \text{for } i = 1, \cdots, n$$

whenever S is any Borel set contained in F. Inasmuch as

$$H_n^{n-1}(N - F) \leqq \frac{\alpha(n)}{4\alpha(n-1)} \mu(N - F) = 0,$$

it follows that if S is any Borel set contained in E_n, then

$$\Phi_i(S) = \Phi_i(S \cap F) = \int_{S \cap F} \nu_i(A, x) dH_n^{n-1} x$$

$$= \int_{S \cap N} \nu_i(A, x) dH_n^{n-1} x = \int_S \nu_i(A, x) dH_n^{n-1} x$$

for $i = 1, \cdots, n$. Accordingly (3) is a consequence of (1).

In the special case when H_n^{n-1} (Boundary A) $< \infty$, the condition (2) is obviously satisfied. Therefore (3) holds in this case, as shown by quite different methods in [**F 1**], [**F 2**], [**F 3**].

Another special case, which includes the preceding one, occurs when the $n-1$ dimensional integralgeometric measure of the boundary of A is finite. Here the validity of (1) may be derived from known characterizations ([**F 4**], [**K**]) of distributions whose partial derivatives are measures.

Finally consider a region A whose boundary B is a finitely triangulable $n-1$ dimensional manifold. In case $n = 2$ the condition (1) holds if and only if the simple closed curve B has finite length. For $n \geqq 3$ the situation is much more complicated. If the inclusion map b of B into E_n has finite $n-1$ dimensional Lebesgue area and if $L_n(B)$

$=0$, then (1) holds. It would be interesting to know whether (1) implies that the Lebesgue area of b is finite.

References

DG 1. E. De Giorgi, *Su una teoria generale della misura* $r-1$ *dimensionale in un spazio ad* r *dimensioni*, Annali di Matematica Ser. 4 vol. 36 (1954) p. 191.

DG 2. ———, *Nuovi teoremi relativi alle misure* $r-1$ *dimensionali in uno spazio ad* r *dimensioni*, Ricerche di Matematica vol. 4 (1955) p. 95.

F 1. H. Federer, *The Gauss-Green Theorem*, Trans. Amer. Math. Soc. vol. 58 (1945) p. 44.

F 2. ———, *Coincidence functions and their integrals*, Trans. Amer. Math. Soc. vol. 59 (1946) p. 441.

F 3. ———, *The* (ϕ, k) *rectifiable subsets of* n *space*, Trans. Amer. Math. Soc. vol. 62 (1947) p. 114.

F 4. ———, *An analytic characterization of distributions whose partial derivatives are representable by measures*, Bull. Amer. Math. Soc. Abstract 60-4-407 (1954).

FY. W. H. Fleming and L. C. Young, *Representations of generalized surfaces as mixtures*, Rend. Circ. Mat. Palermo Ser. 2 vol. 5 (1956) p. 1.

K. K. Krickeberg, *Distributions and Lebesgue area*, Bull. Amer. Math. Soc. Abstract 63-4-437 (1957).

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