

A THEOREM ON QUASI-COMPACT MAPPINGS¹

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1. **Introduction.** The ascertainment of conditions for topological invariances under various types of mappings, especially where the domain space is metric and the mappings are closed or quasi-compact, has received a considerable amount of recent attention. Necessary and sufficient conditions for the invariance of metrizable spaces under closed mappings have been obtained independently by K. Morita and S. Hanai [1] and by A. H. Stone [2]. A similar type theorem for quasi-compact mappings on locally compact, separable metric spaces is given in the present paper. Previous results in this direction have been obtained by A. V. Martin [3], W. E. Malbon [4] and Stone [2].

2. **Definitions.** Let $f(X) = Y$ where X, Y are topological spaces and f is a continuous mapping. Then f is a *quasi-compact mapping* provided that the image of every open inverse set in X is open in Y . The mapping f is a P_1 *mapping* provided that whenever $y \in Y$ and U is a neighborhood of $f^{-1}(y)$ then $y \in \text{int } f(U)$; f is a P_2 *mapping* provided that for each y in Y there is a compact set $C \subset f^{-1}(y)$ such that whenever U is a neighborhood of C then $y \in \text{int } f(U)$. The mapping f is *semi-closed* provided that whenever C is a compact subset of X , then $f(C)$ is closed. The space X is an M *space* provided that whenever $(x_i) \rightarrow x$ and $(x_i) \rightarrow x'$ in X , then $x = x'$.

3. **THEOREM.** *Let f be a quasi-compact mapping of a locally compact, separable metric space X onto the topological space Y ; then Y is a locally compact, separable metric space if and only if f is a semi-closed, P_2 mapping.*

It will be convenient to prove several lemmas before establishing this theorem. It may be observed from the definitions that every P_2 mapping is a P_1 mapping, and every P_1 mapping is quasi-compact. Throughout this paper, f will denote a continuous mapping of a topological space X onto a topological space Y .

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LEMMA 1. *If $f(X) = Y$, then f is P_1 if and only if whenever $y \in \bar{A} - A \subset Y$, then $f^{-1}(y) \cdot \text{Cl}(f^{-1}(A)) \neq \emptyset$.*

PROOF. Let $f(X) = Y$ be P_1 and suppose $y \in \bar{A} - A$ where $f^{-1}(y) \cdot \text{Cl}(f^{-1}(A)) = \emptyset$; then for each x in $f^{-1}(y)$ there is a neighborhood U_x of x such that $U_x \cdot f^{-1}(A) = \emptyset$. Let $U = \sum U_x$, summed over all x in $f^{-1}(y)$; then U is a neighborhood of $f^{-1}(y)$ and y is not interior to $f(U)$, giving a contradiction.

To prove the sufficiency, suppose $y \in Y$, U is a neighborhood of $f^{-1}(y)$, and y is not interior to $f(U)$. If $A = Y - f(U)$, then $y \in \bar{A} - A$ and $f^{-1}(y) \cdot \text{Cl}(f^{-1}(A)) = \emptyset$; this gives a contradiction.

LEMMA 2. *Let $f(X) = Y$ be a P_1 , semi-closed mapping of a metric space X onto Y ; then Y is an M space.*

PROOF. Y is T_1 since f is semi-closed. Let $(y_i) \rightarrow y$ in Y ; it must be shown that if $y' \in Y - y$ then it is false that $(y_i) \rightarrow y'$. By application of Lemma 1 and the metrizable of X , it follows that there is a sequence of points (x_i) in $f^{-1}(\sum y_i - y - y')$ and an x in $f^{-1}(y)$ such that $(x_i) \rightarrow x$. Then $(\sum x_i + x)$ is a compact set and hence $f(\sum x_i + x)$ is a closed infinite subset of $(\sum y_i + y - y')$; therefore $(y_i) \rightarrow y'$ is false.

LEMMA 3. *Let $f(X) = Y$ be a P_2 , semi-closed mapping of a metric space X onto Y ; then Y is a Hausdorff space with a locally countable basis.*

PROOF. Let $y \in Y$ and choose a compact set $C \subset f^{-1}(y)$ such that if U is a neighborhood of C then $y \in \text{int } f(U)$. Let $S_k(C)$ denote the $1/k$ neighborhood of C , and let $W_k = \text{int } f(S_k)$, $k = 1, 2, \dots$. If W is a neighborhood of y and ρ is a metric for X , then $\rho(C, \text{Fr } f^{-1}(W)) > 1/k$ for some positive integer k and hence $W_k \subset W$. This proves (W_k) is a basis at y ; since Y is an M space by Lemma 2, it follows that Y is also Hausdorff [4, 1.3]. For if $(W_i), (W'_i), i = 1, 2, \dots$ were respectively decreasing bases at points y and y' in Y not having disjoint neighborhoods, then by choosing y_i in $W_i \cdot W'_i$ there results a sequence (y_i) such that $(y_i) \rightarrow y, (y_i) \rightarrow y'$ and this gives a contradiction.

The sufficiency of the P_2 and semi-closed conditions in the theorem now follows by Lemma 3 and a theorem proved by A. H. Stone [2; Theorem 3] to the effect that a Hausdorff space with a locally countable basis is a locally compact, separable metric space if it is the image under a quasi-compact mapping of a locally compact, separable metric space.

In regard to the necessity of the P_2 and semi-closed conditions in the theorem, it is evident that f is semi-closed. Suppose f is not a P_2

mapping. Then there is a $y \in Y$ such that for each compact set $C \subset f^{-1}(y)$ there is a neighborhood U of C such that y is not interior to $f(U)$. By application of local compactness and the Lindelöf property, there is a representation $f^{-1}(y) = \sum_{k=1}^{\infty} C_k$, where C_k are compact nonempty sets and $C_{k+1} \supset C_k$. Choose U_1 a conditionally compact neighborhood of C_1 such that y is not interior to $f(U_1)$, and let U_k be inductively defined as a conditionally compact neighborhood of $C_k + f^{-1}(y) \cdot \text{Cl}(U_{k-1})$ such that y is not interior to $f(U_k)$. Let $V_k = \sum_{i=1}^k U_i$, $k=1, 2, \dots$; it will now be shown that y is not interior to $f(V_k)$, $k=1, 2, \dots$. Suppose $y \in \text{int } f(V_k)$ for some k ; then there is a sequence $(y_i) \rightarrow y$ in Y where y_i is in $f(V_k) - f(U_k)$, $i=1, 2, \dots$. If $x_i \in f^{-1}(y_i) \cdot V_k$, the sequence (x_i) has a limit point x in $\text{Cl}(V_{k-1})$ and $f(x) = y$ by the continuity of f . Hence $x \in f^{-1}(y) \cdot \text{Cl}(V_{k-1}) = f^{-1}(y) \cdot \text{Cl}(U_{k-1}) \subset U_k$, contradicting the fact that x is a limit point of (x_i) and x_i is not in U_k , $i=1, 2, \dots$. This proves y is not interior to $f(V_k)$ for $k=1, 2, \dots$.

Since by Lemma 1, Theorem 3 of Stone [2, p. 694], y is interior to $f(V_k)$ for some k , there is a contradiction and therefore f is a P_2 mapping.

The P_1 and P_2 properties of mappings are equivalent whenever all point inverses are compact. Included among the P_1 mappings are all open and all closed mappings [4, pp. 43–45]. For if f is open then f is clearly P_1 ; if f is closed and U is a neighborhood of $f^{-1}(y)$ then $f(X - U)$ is closed and therefore $y \in \text{int}(Y - f(X - U)) \subset \text{int } f(U)$. Hence it is easily seen that all open mappings and all closed mappings with compact point inverses are P_2 mappings.

The following, a modification of an example due to Malbon [4, Ex. 2.28], shows that X must be assumed locally compact in the theorem. In the plane: for $0 < x \leq 1$ let A_x be the line segment $x = x$, $1 \leq y \leq 2$ and p_x the point $(x, 0)$; let q be the point $(0, 1)$. Now let $X = \sum(A_x + p_x) + q$ for $0 < x \leq 1$, with the usual topology of the plane, be decomposed as follows. Elements are p_x and A_x for $x = 1/i$, $i=1, 2, \dots$, $p_x + A_x$ for x otherwise, and q . Let Y denote the induced decomposition space and f the natural map. Then X is separable metric, though not locally compact at q , and f is open, semi-closed, and has compact point inverses. Therefore Y is Hausdorff with a locally countable basis by Lemma 3. If $P = \sum_{i=1}^{\infty} p_{1/i}$, then P is a closed subset of Y and P and q do not have disjoint neighborhoods in Y . Therefore Y is not regular and hence not separable metric.

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CONNECTED SETS OF VAN VLECK

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Both Vitali and Van Vleck have given interesting constructions of Lebesgue nonmeasurable sets in euclidean E_1 . Here we give a generalization for the construction of Van Vleck for E_m , $m \geq 2$; our interest is in the type of connected set that can be so obtained. Elsewhere we will consider the construction of Vitali. Of interest also is the interlacing of these connected sets.

Below Ω is the first transfinite ordinal whose cardinal is the same as that of the linear continuum: α, β, γ are ordinals, > 0 and $< \Omega$. We will say that the *Van Vleck basic set for a given point* $P_\alpha = (x_{1\alpha}, x_{2\alpha}, \dots, x_{m\alpha})$ in (x_1, x_2, \dots, x_m) -coordinate space is the set of all $(x'_{1\alpha}, x'_{2\alpha}, \dots, x'_{m\alpha})$ where for each j ($j=1, 2, \dots, m$), we have as in [1, p. 240],

$$(1) \quad x'_{j\alpha} = \frac{x_{j\alpha}}{2^{(\pm p)}} \pm \frac{u}{2^v} \quad (u, v, p = 0, 1, 2, \dots).$$

Van Vleck constructs two complementary sets in E_1 , each a reflection about $x=1/2$ of the other; it is in part because measure is invariant under reflection that his construction gives Lebesgue nonmeasurable sets.

We will take the *Van Vleck x_1 -reflected set for P_α* as the set of all $(-x'_{1\alpha}, x'_{2\alpha}, \dots, x'_{m\alpha})$ with $x'_{j\alpha}$ as above, i.e. it is the Van Vleck basic set for P_α reflected about $x_1=0$; and the (x_h, x_g, \dots, x_k) -reflected set for P_α is the Van Vleck basic set for P_α reflected about the coordinate subspace where $x_h=0, x_g=0, \dots, x_k=0$, i.e. the set of all $(x''_{1\alpha}, x''_{2\alpha}, \dots, x''_{m\alpha})$, where $x''_{t\alpha} = -x'_{t\alpha}$ for $t=h, g, \dots, k$ and otherwise $x''_{t\alpha} = x'_{t\alpha}$; these reflections include one about the origin, referred to as the $(m-m)$ -coordinate space below.