NOTE ON FORMAL PROPERTIES OF CERTAIN CONTINUED FRACTIONS[1]

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1. **Introduction.** It is the purpose of this note to point out the connection between the work of Bauer [**1**][2] and Frank [**3**] on certain continued fractions. A matrix approach similar to that in [**1**] shows clearly that these are analogues of Stieltjes-type and Jacobi-type continued fractions. An example is given of an expansion used for numerical purposes in [**1**], which is closely related to the Euler expansion [**2**]. This is obtained by a factorization of the Frobenius matrix. Special cases of such expansions coincide with certain cases of the hypergeometric continued fraction of Frank [**3**]. There is also a connection shown between the expansions in §2 and the extended Schur continued fraction [**4**].

2. **Matrix approach.** Let π_n define the space of polynomials of degree n with coefficient of the highest power equal to 1. If one is given a polynomial $P(x)\in\pi_n$ and a polynomial $p_{n-1}(x)\in\pi_{n-1}$, then there are defined polynomials $p_\mu(x)\in\pi_\mu$ $(n-2\geqq\mu\geqq 0)$ and polynomials $r_\mu(x)\in\pi_\mu$ $(n-1\geqq\mu\geqq 0)$ by the euclidean algorithm for $P(x)\equiv r_n(x)$ and $p_{n-1}(x)$, as follows:

$$xp_i(x) - r_{i+1}(x) = q_{n-i}r_i(x), \qquad i = n-1, n-2, \cdots, 0, \tag{2.1}$$

$$r_i(x) - p_i(x) = e_{n-i}p_{i-1}(x), \quad (e_n \equiv 0), i = n-1, n-2, \cdots, 0. \tag{2.2}$$

Define the row-vectors

$$\mathcal{P} = (p_{n-1}(x), p_{n-2}(x), \cdots, p_0(x)), \tag{2.3}$$

$$\mathcal{R} = (r_{n-1}(x), r_{n-2}(x), \cdots, r_0(x)). \tag{2.4}$$

Then the euclidean algorithm takes the form

$$x\cdot\mathcal{P} = \mathcal{R}Q \bmod P(x), \tag{2.5}$$

and

$$\mathcal{R} = \mathcal{P}E, \tag{2.6}$$

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

where Q and E are the "Stieltjes-type" matrices

$$Q = \begin{pmatrix} q_1 & 1 & & & 0 \\ & q_2 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & \ddots \end{pmatrix}, \tag{2.7}$$

$$E = \begin{pmatrix} 1 & & & & 0 \\ e_1 & 1 & & & \\ & e_2 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & & 1 \end{pmatrix}. \tag{2.8}$$

On elimination of either $\mathcal{R}$ or $\mathcal{P}$, one obtains

$$x\mathcal{P} = \mathcal{P}EQ \bmod P(x), \tag{2.9}$$

and

$$x\mathcal{R} = \mathcal{R}QE \bmod P(x), \tag{2.10}$$

where EQ and QE are the tri-diagonal matrices

$$EQ = \begin{pmatrix} q_1 & 1 & & & & & 0 \\ e_1q_1 & e_1 + q_2 & 1 & & & & \\ & e_2q_2 & e_2 + q_3 & 1 & & & \\ & & e_3q_3 & e_3 + q_4 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & 1 \\ 0 & & & & & e_{n-1}q_{n-1} & e_{n-1} + q_n \end{pmatrix}, \tag{2.11}$$

$$QE = \begin{pmatrix} e_1 + q_1 & 1 & & & & & 0 \\ e_1q_2 & e_2 + q_2 & 1 & & & & \\ & e_2q_3 & e_3 + q_3 & 1 & & & \\ & & e_3q_4 & e_4 + q_4 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & 1 \\ 0 & & & & & e_{n-1}q_n & q_n \end{pmatrix}. \tag{2.12}$$

From these formulae one may write down immediately the two J-fractions

$$(2.13)\qquad \frac{p_{n-1}(x)}{P(x)} = \frac{1}{x-q_1-}\,\frac{e_1q_1}{x-q_2-e_1-}\cdots\frac{e_{n-1}q_{n-1}}{x-q_n-e_{n-1}}$$

and

$$(2.14)\qquad \frac{r_{n-1}(x)}{P(x)} = \frac{1}{x-q_1-e_1-}\,\frac{e_1q_2}{x-q_2-e_2-}\cdots\frac{e_{n-1}q_n}{x-q_n}\,.$$

Furthermore, formulae (2.1) and (2.2) can be represented by the matrix equation

$$(2.15)\qquad (\mathfrak{P}\mathfrak{R})\begin{pmatrix} xI & -E \\ -Q & I \end{pmatrix} = 0 \bmod P(x).$$

The compound matrix in this formula may be rearranged to the tridiagonal form

$$\begin{pmatrix} x & -1 & & & & & & & 0 \\ -q_1 & 1 & -1 & & & & & & \\ & -e_1 & x & -1 & & & & & \\ & & -q_2 & 1 & -1 & & & & \\ & & & -e_2 & x & -1 & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & -e_{n-1} & x & -1 & \\ 0 & & & & & & -q_n & 1 & \end{pmatrix}.$$

From this one obtains the Stieltjes continued fraction

$$(2.16)\qquad \frac{p_{n-1}(x)}{P(x)} = \frac{1}{x-}\,\frac{q_1}{1-}\,\frac{e_1}{x-}\cdots\frac{q_n}{1}\,.$$

$\hat{2}$. **A similar approach.** It is possible to set up another division algorithm which differs from the preceding by the fact that the matrix E in (2.6) is not a Stieltjes-type matrix but the reciprocal of such a matrix. In this algorithm, all quantities are denoted by an *accent circonflexe*. One has in this case

$$(\hat{2}.5)\qquad x\cdot\hat{\mathfrak{P}} = \hat{\mathfrak{R}}\hat{Q} \bmod P(x),$$

$$(\hat{2}.6)\qquad \hat{\mathfrak{R}} = \hat{\mathfrak{P}}\hat{E}^{-1},$$

$$(\hat{2}.9)\qquad x\cdot\hat{\mathfrak{P}} = \hat{\mathfrak{P}}\hat{E}^{-1}\hat{Q} \bmod P(x),$$

$$(\hat{2}.10)\qquad x\cdot\hat{\mathfrak{R}} = \hat{\mathfrak{R}}\hat{Q}\hat{E}^{-1} \bmod P(x).$$

Formulae ($\hat{2}$.5) and ($\hat{2}$.6) give explicitly the division algorithm for $P(x) \equiv \hat{r}_n(x)$ and $\hat{r}_{n-1}(x)$, namely,

$$(\hat{2}.1) \qquad \hat{r}_{i+1}(x) + \hat{q}_{n-i}\hat{r}_i(x) = x\hat{p}_i(x), \qquad i = n-1, n-2, \cdots, 0,$$

$$(\hat{2}.2) \qquad \hat{p}_i(x) - \hat{r}_i(x) = \hat{e}_{n-i}\hat{r}_{i-1}(x), \ (\hat{e}_n \equiv 0), i = n-1, n-2, \cdots, 0.$$

From ($\hat{2}$.1) and ($\hat{2}$.2), or from ($\hat{2}$.10) in the form

$$\hat{\mathfrak{R}} \cdot (x\hat{E} - \hat{Q}) = 0 \bmod P(x),$$

one obtains the analogue of (2.14),

$$(\hat{2}.14) \qquad \frac{\hat{r}_{n-1}(x)}{P(x)} = \frac{1}{x - \hat{q}_1} + \frac{\hat{e}_1 x}{x - \hat{q}_2} + \frac{\hat{e}_2 x}{x - \hat{q}_3} + \cdots + \frac{\hat{e}_{n-1} x}{x - \hat{q}_n}.$$

From ($\hat{2}$.1) and ($\hat{2}$.2) with i replaced by $i+1$, one obtains the recurrence relation

$$(\hat{2}.1a) \qquad \hat{p}_{i+1}(x) + (\hat{q}_{n-i} - \hat{e}_{n-i-1})\hat{r}_i(x) = x\hat{p}_i(x), \ (\hat{e}_0 \equiv 0),$$
$$i = n-1, n-2, \cdots, 0.$$

From ($\hat{2}$.1a) and ($\hat{2}$.2) one can then derive the analogue of the Stieltjes-type continued fraction (2.16),

$$(\hat{2}.16) \qquad \frac{\hat{r}_{n-1}(x)}{P(x)} = \frac{1}{-\hat{q}_1} + \frac{x}{1} - \frac{\hat{e}_1}{-\hat{q}_2 + \hat{e}_1} + \frac{x}{1} - \frac{\hat{e}_2}{-\hat{q}_3 + \hat{e}_2} + \frac{x}{1} - \cdots - \frac{\hat{e}_{n-1}}{-\hat{q}_n + \hat{e}_{n-1}} + \frac{x}{1}.$$

Furthermore, one can derive the analogue of the J-fraction (2.13)

$$(\hat{2}.13) \qquad \frac{\hat{p}_{n-1}(x)}{P(x)} = \frac{1}{x - \dfrac{\hat{q}_1}{\hat{q}_2 - \hat{e}_1}\hat{q}_2 + x - \dfrac{\dfrac{\hat{q}_1}{\hat{q}_2 - \hat{e}_1}\hat{e}_1 x}{\dfrac{\hat{q}_2 - \hat{e}_1}{\hat{q}_3 - \hat{e}_2}\hat{q}_3 + x - \dfrac{\dfrac{\hat{q}_2 - \hat{e}_1}{\hat{q}_3 - \hat{e}_2}\hat{e}_2 x}{\dfrac{\hat{q}_3 - \hat{e}_2}{\hat{q}_4 - \hat{e}_3}\hat{q}_4 + \cdots + x - \dfrac{\dfrac{\hat{q}_{n-2} - \hat{e}_{n-3}}{\hat{q}_{n-1} - \hat{e}_{n-2}}\hat{e}_{n-2} x}{\dfrac{\hat{q}_{n-1} - \hat{e}_{n-2}}{\hat{q}_n - \hat{e}_{n-1}}q_n + x - (\hat{q}_n - \hat{e}_{n-1})}}}}.$$

The continued fractions (2.13) and (2.14) are essentially the even and odd parts, respectively, of (2.16). Similarly, the analogues of these continued fractions, namely, ($\hat{2}$.13) and ($\hat{2}$.14), are the odd and even parts, respectively, of ($\hat{2}$.16) (cf. Perron [**5**, pp. 12–13]).

Note added in proof: For completeness we mention the following continued fraction

$$\frac{\hat{p}_{n-1}(x)}{p(x)} = \frac{1}{x} \; \frac{\hat{q}_1\hat{q}_2}{-\hat{q}_2 - \hat{e}_1 +} \; \frac{\hat{e}_1 x}{x} \; \frac{\hat{q}_2\hat{q}_3}{-\hat{q}_3 - \hat{e}_2 +} \; \frac{\hat{e}_2 x}{x} \; - \cdots$$

($\widehat{2}$.13) and ($\widehat{2}$.14) are essentially the even and odd parts, respectively, of this continued fraction.

3. **A continued fraction expansion for** $1/(1+a_1z+a_2z^2+a_3z^3+\cdots)$. The Frobenius matrix

$$F = \begin{pmatrix} -a_1 & 1 & & & 0 \\ -a_2 & & 1 & & \\ -a_3 & & & 1 & \\ \vdots & & & & \ddots \\ -a_{n-1} & & & & 1 \\ -a_n & 0 & & & 0 \end{pmatrix}$$

of the polynomial

$$P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

can be factored into the product $\hat{E}^{-1}\hat{Q}$, namely,

$$F = \begin{pmatrix} 1 & & & & & \\ \frac{-a_2}{a_1} & 1 & & & & \\ & \frac{-a_3}{a_2} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \frac{-a_n}{a_{n-1}} & 1 & \end{pmatrix}^{-1} \cdot \begin{pmatrix} -a_1 & 1 & & & & \\ & \frac{-a_2}{a_1} & 1 & & & \\ & & \frac{-a_3}{a_2} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & \\ & & & & \frac{-a_n}{a_{n-1}} & \end{pmatrix},$$

(cf. Bauer [1, p. 189]). Consequently, one can immediately write a special continued fraction of the form ($\widehat{2}$.14) with $\hat{e}_i = -a_{i+1}/a_i$, $i=1, 2, \cdots, n-1$, and $\hat{q}_i = -a_i/a_{i-1}$, $a_0=1$, $i=1, 2, \cdots, n$. The polynomials $\hat{p}_i(x)$ from which this continued fraction is generated can be calculated from the recurrence formulae ($\widehat{2}$.1) and ($\widehat{2}$.2) where one starts with $r_0 \equiv 1$, $p_0 \equiv 1$. One obtains

$$\hat{p}_i(x) = x^i, \qquad i = 0, 1, \cdots, n-1, \tag{3.1}$$

and

$$\hat{p}_i(x) = \frac{1}{a_{n-i}}(a_{n-i}x^i + a_{n-i+1}x^{i-1} + \cdots + a_n). \tag{3.2}$$

Therefore, the continued fraction expansion for $a_1^{-1}(P(x)-x^n)/P(x)$ is

$$(3.3)\qquad \frac{1}{x+a_1-}\ \frac{\frac{a_2}{a_1}x}{x+\frac{a_2}{a_1}-}\ \frac{\frac{a_3}{a_2}x}{x+\frac{a_3}{a_2}-}\cdots-\frac{\frac{a_n}{a_{n-1}}x}{x+\frac{a_n}{a_{n-1}}},$$

(formula 48 of Bauer [1]), or

$$(3.4)\qquad \frac{x^n}{P(x)}=1-\frac{a_1}{x+a_1-}\ \frac{\frac{a_2}{a_1}x}{x+\frac{a_2}{a_1}-}\cdots-\frac{\frac{a_n}{a_{n-1}}x}{x+\frac{a_n}{a_{n-1}}}.$$

The substitution of $x=1/z$ gives

$$(3.5)\qquad \frac{1}{1+a_1z+a_2z^2+\cdots+a_nz^n}=1-\frac{a_1z}{1+a_1z-}\ \frac{\frac{a_2}{a_1}z}{1+\frac{a_2}{a_1}z-}\cdots-\frac{\frac{a_n}{a_{n-1}}z}{1+\frac{a_n}{a_{n-1}}z}.$$

When one takes the reciprocal of both sides of (3.5) (even applied to an infinite sequence $\{a_\mu\}$), the resulting expansion is the well-known equivalent continued fraction of Euler [2] (with a slight transformation at the beginning),

$$(3.6)\qquad 1+a_1z+a_2z^2+\cdots\equiv\frac{1}{1-}\ \frac{a_1z}{1+a_1z-}\ \frac{\frac{a_2}{a_1}z}{1+\frac{a_2}{a_1}z-}\ \frac{\frac{a_3}{a_2}z}{1+\frac{a_3}{a_2}z-}\cdots.$$

If the series on the left-hand side of (3.6) converges, the equivalent continued fraction on the right-hand side converges to the same value, while, if the series diverges, the continued fraction likewise diverges.

4. **Other special cases.** For the special case of the hypergeometric function $F(\alpha, 1, \gamma, z)$, (3.6) takes the form

$$(4.1)\qquad \frac{1}{F(\alpha,1,\gamma,z)} = 1 - \frac{\dfrac{\alpha}{\gamma}z}{1+\dfrac{\alpha}{\gamma}z} - \frac{\dfrac{\alpha+1}{\gamma+1}z}{1+\dfrac{\alpha+1}{\gamma+1}z} - \cdots,$$

valid for $|z|<1$ (cf. Frank [**3**, formula 5.2]). For $|z|>1$, (4.1) converges to the value 0 [**3**, Theorem 3.1].

Furthermore, expansions ($\hat{2}$.13), ($\hat{2}$.14), and ($\hat{2}$.16) can be compared with the generalized Schur continued fractions (cf. Frank [**4**]). The expansion (1.1) of [**4**],

$$(4.2)\qquad \frac{k_0(1-\gamma_0\bar{\gamma}_0)z}{\bar{\gamma}_0 z} \quad \frac{1}{-k_1\gamma_1+} \quad \frac{k_1(1-\gamma_1\bar{\gamma}_1)z}{\bar{\gamma}_1 z} \quad \frac{1}{-k_2\gamma_2+}\cdots, \qquad |\gamma_i|\neq 1,$$

is equivalent to the expansion ($\hat{2}$.16) for $\hat{f}_{n-1}(1/z)/P(1/z)$ if the k_i and γ_i have the values

$$(4.3)\qquad \begin{aligned} &k_0(1-\gamma_0\bar{\gamma}_0) = -1, \qquad k_i(1-\gamma_i\bar{\gamma}_i) = \hat{e}_i,\\ &\bar{\gamma}_0 = \hat{q}_1, \quad \bar{\gamma}_i = \hat{q}_{i+1} - \hat{e}_i, \quad i = 1, 2, \cdots, n-1, \end{aligned}$$

with the additional restriction

$$(4.4)\qquad k_i\gamma_i = 1, \qquad i = 1, 2, \cdots, n.$$

With the same values (4.3) and (4.4), the even part of (4.2) (cf. expansion (6.1) of [**4**]) is equivalent to the expansion ($\hat{2}$.14) for $\hat{f}_{n-1}(1/z)/P(1/z)$. Also, the odd part of (4.2) is equivalent with the values (4.3) and (4.4) to the expansion for $\hat{f}_{n-1}(1/z)/P(1/z)$ which can be obtained from the value for $\hat{p}_{n-1}(1/z)/P(1/z)$ from ($\hat{2}$.13), as follows:

$$\frac{\hat{f}_{n-1}(1/z)}{P(1/z)} = -\frac{1}{\hat{q}_1} + \frac{1}{\hat{q}_1 z}\cdot\frac{\hat{p}_{n-1}(1/z)}{P(1/z)}.$$

For these continued fractions, convergence regions are found in [**4**].

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A THEOREM ON A-LOOPS

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In a recent article, R. H. Bruck and Lowell J. Paige have investigated A-loops, or loops for which every element of the inner mapping group is an automorphism (see [1]). Specifically, they have shown that A-loops with the inverse property are diassociative and that there exist noncommutative diassociative A-loops. The authors also conjecture that the only commutative diassociative A-loops are the commutative Moufang loops. The purpose of the present note is to offer a proof of this conjecture.

Let y and z be two elements of a commutative diassociative A-loop G, and let R_y denote right multiplication in G by the element y. Then $S=R_zR_yR_{zy}^{-1}$ is an element of the inner mapping group, so that $wS\cdot xS=(wx)S$ for every pair of elements w and x of G. Setting $w=pq$ and $x=q^{-1}$ gives $(pq)S=pS\cdot[q^{-1}S]^{-1}$, and comparing with the first equation, we see that

$$xR_zR_yR_{zy}^{-1} = xS = [x^{-1}S]^{-1} = [(x^{-1}z\cdot y)\cdot(zy)^{-1}]^{-1}$$
$$= (xz^{-1}\cdot y^{-1})\cdot(zy) = xR_z^{-1}R_y^{-1}R_{zy}\,.$$

Thus[1] $R_zR_yR_{zy}^{-1}=R_z^{-1}R_y^{-1}R_{zy}$, or $R_{zy}^2=R_yR_z^2R_y$. Using diassociativity, we can write this as $(xy\cdot z^2)\cdot y=x\cdot(y\cdot z^2y)$, which is just a form of the Moufang identity except for the fact that z is squared. Our problem is to show that the identity holds without this restriction.

We observe, first of all, that the subloop of G consisting of all squares is a commutative Moufang loop. This already proves our theorem for loops all of whose elements have odd orders. One expects difficulty for loops containing elements of order two, particularly

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[1] This formula may also be obtained from equation (3.24) of [1] by setting $L_z=R_z$.