ON THE ITERATION OF TRANSFORMATIONS IN NONCOMPACT MINIMAL DYNAMICAL SYSTEMS

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Let A be a Hausdorff space, ϕ a continuous mapping of A into itself. It is the purpose of the present paper to discuss various topics centering around the following question: If g is a bounded continuous function on A, does there exist a bounded continuous function f on A such that $f(\phi a)-f(a)=g(a)$ for all a in A? Suppose that for each a_0 in A, the set $\{\phi^n a_0,\ n\geqq 0\}$ is dense in A. Theorem 1 asserts that a necessary and sufficient condition for the existence of such an f is that $\left|\sum_{k=0}^{j} g(\phi^k a)\right|$ should be uniformly bounded for all positive j and all points a of A. For homeomorphisms of compact spaces, this result was previously obtained by Gottschalk and Hedlund [5, Theorem 14.11, p. 135].[1]

A related problem for linear operators in a Banach space is obtained by letting X be the Banach space of bounded continuous functions on A with the uniform norm, T the linear transformation of X into itself defined by $(Tf)(a)=f(\phi a)$, $a\in A$. In terms of X and T, Theorem 1 states that g will lie in the range of $(I-T)$ if and only if the sequence of norms $\left\|\sum_{k=0}^{j} T^k g\right\|$ is uniformly bounded for all positive j. In a reflexive Banach space, this characterization of the range of $(I-T)$ is valid for any linear transformation T for which $\|T^n\|$ is bounded for all n. A sufficient condition in a general Banach space would seem to require an assumption that the elements $\{\sum_{k=0}^{j} T^k g\}$ lie for all j in a fixed weakly compact subset K of X. It would be interesting to obtain a proof of Theorem 1 along these lines. We shall content ourselves with showing by these methods that if m is a totally-finite measure on a σ-algebra on A, $L^\infty(m)$ the space of m-essentially bounded measurable functions, ϕ a measure preserving mapping of A into A, then in order that for an element g in $L^\infty(m)$, there should exist an f in $L^\infty(m)$ such that $f(\phi a)-f(a)=g(a)$ a.e. in m, it is necessary and sufficient that m-ess. sup. $\left|\sum_{k=0}^{j} g(\phi^k a)\right|$ should be uniformly bounded for all positive j.[2]

Such a result raises another sort of question. For a topological space A if g is continuous and f is a solution of the equation $f(\phi a)$

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[1] The writer's attention was drawn to this question by reading a preprint of [3] in which a theorem of this type is proved for rotations of the circle. This case was already treated by Hedlund in [6, Theorem 3.1, p. 557].

[2] A result in L^∞ for rotations of the circle was already proved by J. Wermer. (Cf. footnote, p. 557 of [6].)

$-f(a)=g(a)$, with f lying in some larger class of functions, must f be necessarily continuous after change on some negligible set? We place the question in a more definite setting. Let A_1 be a Hausdorff space, A_2 a compact topological group, ϕ a homeomorphism of A_1 onto itself such that A_1 is a minimal orbit closure under ϕ, ψ_0 a continuous map of A_1 into A_2. Suppose there exists a Baire function h from A_1 to A_2 satisfying the relation

$$h(\phi a_1) = \psi_0(a_1)\cdot h(a_1), \quad (\cdot, \text{ the group product}), \tag{1}$$

for all a_1 outside some set of the first category in A_1. Then if A_1 is a Baire space, h is continuous after change on a set of the first category.

A similar result is valid if A_2 is merely a compact space, ψ_0 a homeomorphism of A_2 onto itself which generates an equicontinuous transformation group of A_2, and (1) is replaced by

$$h(\phi a_1) = \psi_0(h a_1). \tag{1$'$}$$

In this form, the result has been established by S. Kakutani in [**7**] by rather different methods. One interesting feature of the present proof is that it is valid also under the following hypotheses: A_1 a measure space with a measure m such that all open sets are measurable and have positive measure, ϕ maps null sets on null sets, h a function from A_1 to A_2 continuous on the complement of a set of zero measure. Then h is continuous on the whole of A_1 after replacement on a set of measure zero. An extension is given for functional equations of a more general type than (1)$'$.

1: Let A be a Hausdorff space, ϕ a continuous mapping of A into itself. We assume that A is a minimal orbit closure under ϕ, i.e., for every a_0 in A, the closure of the set $\{\phi^n a_0, n \geqq 0\}$ coincides with A. Let B be another Hausdorff space, ψ a continuous mapping of the Cartesian product $A\times B$ into B.

We define a continuous mapping π of $A\times B$ into itself by setting $\pi(a, b)=(\phi a, \psi(a, b))$. If π^n is the nth iterate of π, $O(a, b)$ is the orbit of (a, b) under the mapping π, i.e. $O(a, b)=\bigcup_{n\geqq 0}\{\pi^n(a, b)\}$, then we let $F(a, b)$ be the closure of $O(a, b)$ in $A\times B$. Let p_A and p_B be the projection mappings of $A\times B$ on its first and second components respectively, $p_A(a, b)=a$, $p_B(a, b)=b$. We shall assume in the following that for each point (a, b) in $A\times B$, $p_B(F(a, b))$ is contained in a compact subset of B.

Consider the family J of subsets of $A\times B$, where $J=\{F \mid F$ is a nonempty closed subset of $A\times B$; $(a, b)\in F$ implies that $\pi(a, b)\in F$; $p_B(F)$ is contained in a compact subset of $B\}$. Since for any point

(a_0, b_0) in $A \times B$, $F(a_0, b_0)$ is an element of J, J is certainly not vacuous.

LEMMA 1. *If $F \in J$, then $p_A(F) = A$.*

PROOF. Let (a_0, b_0) be a point of F. Since $\pi^n(a_0, b_0) \in F$, $p_A \pi^n(a_0, b_0) \in p_A(F)$. Thus $p_A(F)$ contains the dense set $\{\phi^n a_0\}$ and therefore is dense in A. On the other hand, F is closed in $A \times B$, $F \subset A \times \mathrm{Cl}(p_B(F))$, and $\mathrm{Cl}(p_B(F))$ is compact in B. Therefore, $p_A(F)$ is closed in A [1, Exercise 8, p. 68]. Since $p_A(F)$ is dense and closed in A, $p_A(F) = A$.

LEMMA 2. *J has a minimal element under inclusion. Every orbit closure $F(a, b)$ contains a minimal element of J.*

PROOF. By the Lemma of Zorn, it suffices to prove that every subfamily of J which is linearly ordered with respect to inclusion has a lower bound in J. Let $L = \{F_\alpha\}$ be such a family. Then $F_0 = \bigcap_\alpha F_\alpha$ is a closed invariant set under π while $p_B(F_0)$ is certainly contained in a compact subset of B. To prove that $F_0 \in J$, we must show $F_0 \neq \phi$. Let a_0 be a point of A, $G_\alpha = F_\alpha \cap p_A^{-1}(a_0)$. By Lemma 1, G_α is a family of closed sets in $A \times B$ such that every finite subfamily has a nonempty intersection. Moreover, each G_α is a closed subset of $a_0 \times \mathrm{Cl}(p_B(F_\alpha))$, which is compact since it is mapped homeomorphically by p_B on the compact set $\mathrm{Cl}(p_B(F_\alpha))$. Since all the G_α are compact, $G_0 = \bigcap_\alpha G_\alpha$, is nonempty, and, since $G_0 \subset F_0$, F_0 is nonempty.

Let ζ be a homeomorphism of B onto itself commuting with ψ, i.e. such that $\psi(a, \zeta b) = \zeta\psi(a, b)$ for all $a \in A$, $b \in B$. Let S_ζ be the homeomorphism of $A \times B$ onto itself defined by $S_\zeta(a, b) = (a, \zeta b)$.

LEMMA 3. *Let F_0 be a minimal element of J and suppose that for a fixed point a in A, the points (a, b) and (a, b_1) lie in F. Suppose further that there exists a homeomorphism ζ of B onto B commuting with ψ, such that $\zeta b = b_1$. Then $S_\zeta F_0 = F_0$.*

PROOF. From the fact that ζ commutes with ψ, we see that $S_\zeta \pi(a, b) = (\phi a, \zeta\psi(a, b)) = (\phi a, \psi(a, \zeta b)) = \pi S_\zeta(a, b)$. Thus $S_\zeta \pi^n = \pi^n S_\zeta$, and $S_\zeta(O(a, b)) = O(a, \zeta b)$. Since S_ζ is a homeomorphism, $S_\zeta F(a, b) = F(a, \zeta b)$. Since F_0 is a minimal element of J, $F_0 = F(a, b) = F(a, b_1)$. But $S_\zeta F_0 = S_\zeta F(a, b) = F(a, \zeta b) = F(a, b_1) = F_0$.

THEOREM 1. *Let ϕ be a continuous mapping of the Hausdorff space A into itself with A a minimal orbit closure under ϕ. Let g be a bounded continuous function from A to the n-dimensional Euclidean space R^n. In order that there should exist a bounded continuous function f from A to R^n such that $f(\phi a) - f(a) = g(a)$ for all a in A, it is necessary and* *sufficient that there exist a constant $M>0$ with*

$$\sup_{a\in A}\left|\sum_{k=0}^{j} g(\phi^k a)\right| \leqq M \text{ for all } j \geqq 0. \tag{2}$$

PROOF OF THEOREM 1. Necessity is obvious for if $g(a)=f(\phi a)-f(a)$, then $\left|\sum_{k=0}^{j} g(\phi^k a)\right| = |f(\phi^{j+1}a)-f(a)| \leqq 2\sup |f(a)|$.

To prove sufficiency, we specialize our preceding discussion by taking $B=R^n$ and setting $\psi(a, r)=r+g(a)$ for $a\in A$, $r\in R^n$. The corresponding mapping π is defined by $\pi(a, r)=(\phi a, r+g(a))$. The condition (2) is equivalent to the fact that the orbit of any point (a, r) under π has a bounded and hence precompact image in R^n under the projection map p_{R^n} of $A\times R^n$ into R^n. Hence the conclusions of Lemmas 1, 2, and 3 are valid for this mapping π. Let F_0 be a minimal closed invariant set in $A\times R^n$ with respect to π. Suppose that for some point a in A, $p_A^{-1}(a)\cap F_0$ contained two distinct points (a, r), (a, r_1). Let $\xi=r-r_1$, ζ_ξ the homeomorphism of R^n onto itself defined by $\zeta_\xi(r)=r+\xi$. Then ζ_ξ commutes with ψ, $\zeta_\xi(r_1)=r$, and Lemma 3 is applicable. Thus if $S_{\zeta_\xi}(a, r)=(a, r+\xi)$, $S_{\zeta_\xi}F_0=F_0$. But then $S_{\zeta_\xi}^m F_0=F_0$ for any positive integer m, contradicting the boundedness of the second component for elements of F_0. Thereby, we have shown that F_0 has at most one point (a, r) for a given $a\in A$.

Let f be the function from A to R^n defined uniquely by the condition $(a, f(a))\in F_0$. By Lemma 1, f is defined on all of A. f can be considered as a function from A to the compact set $\mathrm{Cl}(p_{R^n}F_0)$. Since F_0, the graph of f, is closed, f is continuous [**1**, Exercise 12, p. 68]. Since $\pi(a, f(a))\in F_0$, we have $(\phi a, f(a)+g(a))\in F_0$, i.e. $f(\phi a)=f(a)+g(a)$.

REMARK. Following a remark of Kakutani, we note that the existence of a minimal subset in J under the hypotheses of Theorem 1 can be proved in an elementary way without the use of Zorn's Lemma or the Axiom of Choice. Let $F_0=F(a_0, r_0)$ for a fixed element (a_0, r_0) in $A\times R^n$. We shall show that F_0 is a minimal element of J. It suffices to show that if $(a, r)\in F_0$, then $(a_0, r_0)\in F(a, r)$. We note first that if $(a_0, r_1)\in F_0$, and $\xi=r_1-r_0$, then $S_{\zeta_\xi}F_0=S_{\zeta_\xi}F(a_0, r_0)=F(a_0, r_1)\subset F_0$. If $\xi\neq 0$, then $S_{\zeta_\xi}^m F_0\subset F_0$ for all $m>0$, contradicting (2). Thus $\xi=0$ and (a_0, r_0) is the only point in $p_A^{-1}(a_0)\cap F_0$. But $p_A^{-1}(a_0)\cap F(a, r)$ is contained in $p_A^{-1}(a_0)\cap F_0$ and is nonempty by Lemma 1. It follows that $(a_0, r_0)\in F(a, r)$ and F_0 is minimal.

2. Let X be a Banach space, T a continuous linear transformation of X into itself.

LEMMA 4. *A sufficient condition for g in X to lie in the range of*

$(I-T)$ is that the set of elemenets $\{\sum_{k=0}^{j} T^k g\}$ should lie for $j\geqq 0$ in a fixed weakly compact subset K of X.

PROOF. By theorems of Eberlein and M. Krein (cf. [4]), the convex closure K' of $K\cup\{0\}$ is weakly sequentially compact. By the principle of uniform boundedness, there exists $M>0$ such that $\|\sum_{k=0}^{j} T^k g\|\leqq M$ for $j\geqq 0$. Thus if we set $g_n=g-n^{-1}\sum_{j=0}^{n-1} T^k g$, then g_n will converge strongly to g as $n\to\infty$. Furthermore, each g_n lies in the range of $(I-T)$ since $g_n=(I-T)\{n^{-1}\sum_{k=1}^{n-1}(\sum_{j=0}^{k-1} T^j g)\}$. Let us set $h_k=\sum_{j=0}^{k-1} T^j g$, $f_n=\{\sum_{k=1}^{n-1} h_j\}\cdot n^{-1}$. Then $g_n=(I-T)f_n$, while the f_n lie for all n in the weakly sequentially compact set K'. Choose a subsequence f_{n_i} converging weakly to an element f of X as $i\to\infty$. Then $(I-T)f_{n_i}$ converges weakly to $(I-T)f$. But $g_{n_i}=(I-T)f_{n_i}$ converges strongly to g. Hence $g=(I-T)f$.

LEMMA 5. *Let X be a reflexive Banach space, T a continuous linear transformation of X into itself. A sufficient condition that g lie in the range of $(I-T)$ is that $\|\sum_{k=0}^{j} T^k g\|$ be uniformly bounded for $j\geqq 0$. If $\|T^n\|\leqq M'$ for $n\geqq 0$, the condition is also necessary.*

PROOF. The necessity is obvious, since if $g=(I-T)f$, $\|\sum_{k=0}^{j} T^k g\| =\|f-T^{j+1}f\|\leqq 2M'$. Sufficiency follows from Lemma 4 since every closed ball about zero in a reflexive space is weakly compact.

THEOREM 2. *Let A be a measure space with a totally finite measure m, ϕ a measure preserving mapping of A into A. In order that for a function g in $L^\infty(m)$, there should exist an $f\in L^\infty(m)$ such that $f(\phi a) -f(a)=g(a)$ a.e. in m, it is necessary and sufficient that*

$$m\text{-ess. sup.}\left|\sum_{k=0}^{j} g(\phi^k a)\right|$$

should be uniformly bounded for $j\geqq 0$.

PROOF OF THEOREM 2. Choose a value of p, $1<p<\infty$. Let T mapping $L^p(m)$ into itself be defined by $(Tf)(a)=f(\phi a)$, $a\in A$. Then $\|Tf\|_{L^p}=\|f\|_{L^p}$, while $\|f\|_{L^p}\leqq m(A)^{1/p}\|f\|_{L^\infty}$ for $f\in L^p\cap L^\infty$. Since necessity is obvious, we consider only sufficiency. Let g be our given function from L^∞. Since $\|\sum_{k=0}^{j} T^k g\|_{L^p}\leqq m(A)^{1/p}\|\sum_{k=0}^{j} T^k g\|$ which is uniformly bounded for $j\geqq 0$, applying Lemma 5 to the reflexive space $L^p(m)$, we conclude that there exists $f_0\in L^p(m)$ such that $f_0(\phi a) -f_0(a)=g(a)$. Since the mean ergodic theorem holds for T in the reflexive space $L^p(m)$, [8] the ergodic means $n^{-1}\sum_{j=0}^{n-1} T^j f_0$ converges to an element f_1 of $L^p(m)$ in the strong topology of $L^p(m)$ and $(I-T)f_1=0$. Let $f=f_0-f_1$. Then $f(\phi a)-f(a)=g(a)$ a.e. while $n^{-1}\sum_{j=0}^{n-1} T^j f\to 0$ in $L^p(m)$ as $n\to\infty$. Set $h_n=-n^{-1}\sum_{j=1}^{n}\sum_{k=0}^{j-1} T^k g$.

Then $\|h_n\|_{L^\infty}$ are uniformly bounded while $h_n = f - n^{-1}\sum_{j=1}^{n} T^j f$ converges in $L^p(m)$ to f as $n\to\infty$. Choosing a subsequence which converges to f a.e., it follows that $f\in L^\infty(m)$.

3. Let A_1 and A_2 be two Hausdorff spaces, with A_1 a Baire space, i.e. of the second category on itself. Let ϕ be a homeomorphism of A_1 onto itself such that A_1 is a minimal orbit closure under ϕ. Let ψ be a continuous mapping from $A_1\times A_2$ into A_2. We shall consider functions h from A_1 to A_2 which satisfy the condition

$$h(a_1) = \psi(a_1, h(a_1)), \qquad a_1 \in A_1. \tag{3}$$

The function h will be said to be a Baire function if there exists a set S of the first category in A_1 such that h is a continuous mapping of $A_1 - S$ into A_2. If A_2 is a metric space, this definition includes all functions obtained by a sequence of pointwise sequential limits starting with continuous functions [**2**, Exercise 14, p. 81].

A family H of homeomorphisms of A_2 is said to be universally transitive if for every pair of distinct points a_2, a_2' in A_2 there is a ζ in H such that $\zeta a_2 = a_2'$.

THEOREM 3. *Let h be a Baire function from A_1 to A_2 for which* (3) *holds outside some set S_1 of first category in A_1. Suppose that A_2 is compact and that there exists a universally transitive family H of homeomorphisms of A_2, each of which has no fixed points and commutes with ψ. Then after change on a set of the first category in A_1, h can be made into a continuous function from A_1 to A_2 satisfying* (3) *for all a_1 in A_1.*

PROOF. Let $S_0 = \bigcup_{n\geq 0}\{\phi^n(S)\cup\phi^n(S_1)\}$. Since ϕ is a homeomorphism, S_0 is of first category in A_1. $A_1 - S_0$ is an invariant set with respect to ϕ and dense in A_1, h is continuous from $A_1 - S_0$ to A_2, and (3) holds for all a_1 in $A_1 - S_0$. If we set $B = A_2$ in the discussion of §1 and $\pi(a_1, a_2) = (\phi a_1, \psi(a_1, a_2))$, the results of Lemmas 1, 2, and 3 are valid for π. Let a_1' be a point of $A_1 - S_0$, $a_2' = h(a_1')$, F_0 a minimal invariant set contained in $F(a_1', a_2')$. The condition (3) on h in $A_1 - S_0$ implies since h is continuous on $A_1 - S_0$, that if G is the graph of h on $A_1 - S_0$, then $G = F(a_1', a_2')\cap p_{A_1}^{-1}(A_1 - S_0)$. We shall show that $F_0 = F(a_1', a_2')$ and that for each a_1 in A_1, $p_{A_1}^{1}(a_1)\cap F_0$ consists of a single point. The function f whose graph is F_0 will then be the desired continuous extension of h.

It suffices to show that if (a_1, a_2) and (a_1, a_2^*) lie in F_0, then $a_2 = a_2^*$. If not, there is a homeomorphism $\zeta\in H$ commuting with ψ and without fixed points on A_2 such that $\zeta a_2 = a_2^*$. By Lemma 3, however $S_\zeta F_0 = F_0$. Since ζ has no fixed points, S_ζ has no fixed points. But then F_0 and a fortiori $F(a_1', a_2')$ would have at least two points over every point of A_1. Since over the points of $A_1 - S_0$, it has only one point, this is impossible.

We may specialize Theorem 3 in two ways: (1) by letting A_2 be a compact group, ψ_0 a mapping of A_1 into A_2, $\psi(a_1, a_2) = \psi_0(a_1) \cdot a_2$, the homeomorphism family H be the elements of $A_2 - \{e\}$ acting by right multiplication on A_2; (2) by letting A_2 be a compact space, ψ_0 be a homeomorphism of A_2 onto itself such that the group generated by ψ_0 is equicontinuous, $\psi(a_1, a_2) = \psi_0(a_2)$, H the closure of the group of homeomorphisms generated by ψ_0 except for the identity. In this second case we may replace A_2 by the orbit closure under ψ_0 of one of the values taken by h on an element of $A_1 - S_0$. It is known [**5**, 9.33, pp. 78–79] that on this orbit closure H is universally transitive and, unless the orbit closure is finite, the elements of H have no fixed points on this set. If we modify h to make it a continuous mapping into this set, it will be a continuous mapping into A_2.

In these two cases the specialized forms of Theorem 3 become:

THEOREM 4. *Let A_1 be a Baire space, A_2 a compact group, ϕ a homeomorphism of A_1 onto itself such that A_1 is a minimal orbit closure under ϕ. Let ψ_0 be a continuous mapping of A_1 into A_2. Suppose that the Baire function h satisfies the relation*

$$h(\phi a_1) = \psi_0(a_1) \cdot h(a_1) \tag{1}$$

for all a_1 outside a set of the first category in A_1. Then after change on a set of the first category in A_1, h can be made into a continuous function from A_1 to A_2 which satisfies (1) *for all $a_1 \in A_1$.*

THEOREM 5. *Let A_1 be a Baire space, A_2 a compact space, ϕ a homeomorphism of A_1 onto itself under which A_1 is a minimal orbit closure, ψ_0 a homeomorphism of A_2 into itself which generates an equicontinuous group of homeomorphisms of A_2. Suppose that the Baire function h from A_1 to A_2 satisfies the relation*

$$h(\phi a_1) = \psi_0(h a_1) \tag{1$'$}$$

for all a_1 outside a set of the first category in A_1. Then after change on a set of the first category in A_1, h can be made into a continous function from a_1 to a_2 satisfying (1)′ *for all a_1 in A_1.*

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ON SPACES WHICH ARE NOT OF COUNTABLE CHARACTER

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It is well known that the unit interval I has a countable base and the fixed point property. By considering the maps $g(x)=x^2$ and $h(x)=1-x$, one sees that there is no $x\in I$ such that for every continuous map $f\colon I\to I$, $x\in f(I)$ implies $f(x)=x$.

In Theorem 1, it is shown that if A is a closed, non-null proper subset of a locally connected, compact Hausdorff space X which has a countable base, then there exists a continuous map $f\colon X\to X$ such that $A\cap f(X)$ is not contained in $A\cap f(A)$. Theorem 2 shows that certain nondegenerate topological spaces X contain proper subsets M such that for every continuous map $f\colon X\to X$, $M\cap f(X)\subset M\cap f(M)$. That is, for each of these spaces X and every continuous map $f\colon X\to X$, $x\in M\cap f(X)$ implies $f^{-1}(x)\cap M\neq\emptyset$. The corollary is of interest in that, if X satisfies the hypotheses of Theorem 2 and M consists of a single point, then a fixed point of some of the maps $f\colon X\to X$ is located.

THEOREM 1. *Suppose X is a connected, locally connected, compact Hausdorff space which has a countable base. If A is any non-null, closed, proper subset of X, then there exists a continuous map $f\colon X\to X$ such that $A\cap f(X)\setminus A\cap f(A)\neq\emptyset$.*

PROOF. Since X is compact Hausdorff and has a countable base, X is metrizable. Hence X is arcwise connected. Let $y\in X\setminus A$. Since

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