LIMIT SECTIONS AND UNIVERSAL POINTS OF CONVEX SURFACES

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1. The following question, raised originally by S. Mazur, appears in S. M. Ulam's collection of mathematical problems [1]: does there exist a closed convex surface whose plane sections give all plane closed convex curves, up to affinities? While this problem is apparently still unsolved the answer is almost certainly negative. At least three different extensions of the problem could be considered: (1) to allow the plane sections an equivalence up to a larger class, or perhaps a larger group, of transformations than the affinities, (2) to ask that the set of all plane sections of the surface should only contain a sufficiently large subset of the set of all closed convex curves, for instance, the set of all convex polygons of given diameter or perimeter, all analytic ovals of fixed length, or all ovals of given constant width, and (3) to generalize the concept of a plane section.

This note is concerned with the last possibility. Instead of plane sections of a surface one considers its limit sections. Roughly speaking, these are limits of sequences of magnified sections of a surface by sequences of planes converging to a supporting plane. For instance, if a strictly convex surface is sufficiently regular the limit section will always be an ellipse with axes proportional to the square roots of the principal radii of curvature. Thus the limit section is a generalization of Dupin's indicatrix.

The following notation will be used: C and D will denote curves, other capital letters will usually denote surfaces, P will be reserved for planes, small letters will stand for points, and small Greek letters will be non-negative constants. A surface (curve) will always mean a closed strictly convex surface (a closed plane convex curve). A part of a surface cut off by a plane will be called a cap. The set of all curves will be denoted by $\mathfrak{U}$.

2. Let S be a surface and let $OXYZ$ be a Cartesian frame with the origin O inside S. Let a sequence $\{P_n\}$ of planes converge to a supporting plane P of S at s. Suppose that all planes P_n intersect S and let $C_n = S \cap P_n$. Let $\{\lambda_n\}$ be a sequence of constants and let $\lambda_n C_n$ denote the curve similar to C_n in the ratio $\lambda_n : 1$. The limit section C of S at s with respect to the sequence $\{P_n\}$ is defined as the limit,

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if it exists, of the sequence $\{\lambda_n C_n\}$. More accurately, it is the limit of suitable reorientations in space, by rigid motions, of the curves of the sequence $\{\lambda_n C_n\}$. It must be emphasized that a limit section depends on the choice of the sequence of the intersecting planes. Further, if C is a limit section, so is λC. The set of all limit sections of a surface at a point s on it will be denoted by $\mathcal{C}(s)$. The set of all parallel limit sections at s, that is, limit sections formed with respect to sequences of planes parallel to a supporting plane at s, will be denoted by $\mathcal{C}_p(s)$. A point s of a surface is called universal if $\mathcal{C}(s)=\mathfrak{U}$, and it is called p-universal if $\mathcal{C}_p(s)=\mathfrak{U}$. A surface S is universal if $\bigcup_{s\in S}\mathcal{C}(s)=\mathfrak{U}$ and it is p-universal if $\bigcup_{s\in S}\mathcal{C}_p(s)=\mathfrak{U}$.

3. THEOREM 1. *There exists a surface S with these properties*: (1) *S is of class C^∞ except at one point s*, (2) *S possesses a unique supporting plane at s*, (3) *s is a p-universal point.*

Select in $\mathfrak{U}$ a countable basis of analytic curves $\{C_n\}$, $n=1, 2, \cdots$. Let $OXYZ$ be a Cartesian coordinate frame, let $\{\alpha_n\}$ be an increasing convergent sequence with $\alpha_0=0$, and let $\{\mu_n\}$ be a decreasing sequence with $\mu_0=1$ and $\lim \mu_n=0$. These sequences will be determined in the process of construction. Let C_0 be the unit circle about the origin in the plane $z=\alpha_0$. In the plane $z=\alpha_n$ place the curve $\mu_n C_n$ so that the following conditions are satisfied: $\mu_n C_n$ encircles the z-axis, the projection of $\mu_{n+1}C_{n+1}$ onto the plane $z=\alpha_n$ lies within $\mu_n C_n$, and any cone with vertex on $\mu_{n+1}C_{n+1}$ and with $\mu_n C_n$ for directrix contains $\mu_0 C_0, \cdots, \mu_{n-1}C_{n-1}$ in its interior. The sequences $\{\alpha_n\}$ and $\{\mu_n\}$ can always be found so that the above conditions are satisfied. It follows now that there exists a cap T with these properties: T is based on C_0, its intersection with the plane $z=\alpha_n$ is $\mu_n C_n$, and it is of class C^∞ at all points except at the point s with coordinates $(0, 0, \lim \alpha_n)$. The last property calls for the standard technique which utilizes the functions like $f(x)=\exp(-1/x^2)$. In addition, by making the sequence $\{\alpha_n\}$ converge fast enough the cap T can be made to possess a unique supporting plane at s.

Now complete T to a surface by closing it up with a hemisphere and modify this surface along the join to a surface S which is of class C^∞ everywhere except at s. The surface S is easily shown to satisfy the conditions of the theorem: for any C in $\mathfrak{U}$ there exists a subsequence $\{C_{n_j}\}$ whose limit is C; now the planes whose equations are $z=\alpha_{n_j}$, together with the sequence $\{\mu_{n_j}^{-1}\}$ of constants, determine the parallel limit section C at s.

THEOREM 2. *There exists a surface S every point of which is universal.*

It will be shown first that it suffices to construct S so that the set of its p-universal points is dense on it. Let S be such a surface and let s be any point on it. By assumption, there exists a sequence $\{s_n\}$ of p-universal points on S, which converges to s. By definition, for each point s_n there exists a convergent sequence $\{P_{mn}\}$ of planes, $m=1, 2, \cdots$, which converges to a supporting plane at s_n, and which determines, together with a sequence $\{\lambda_{mn}\}$ of constants, a parallel limit section C at s_n. Here C is an arbitrary member of $\mathfrak{U}$. Now it is easy to show that the diagonal sequence $\{P_{nn}\}$, together with $\{\lambda_{nn}\}$, determines a limit section C at s.

The surface S with a dense set of p-universal points will be constructed by the process of successive approximations. Let $OXYZ$ be a Cartesian coordinate frame and let S_1 be the unit sphere about the origin. Let $\{H_n\}$ be a sequence of half-rays through O, which is dense in the set of all half-rays through O. Let $\{\beta_n\}$ be a sequence of small positive constants, to be determined later on. Let $s_1=S_1\cap H_1$ and let v_1 be the point on Os_1 whose distance from s_1 is β_1. Put a plane P_1 through v_1 at right angles to Os_1 and remove from S_1 the cap based on P_1. Let $D_1=S_1\cap P_1$ and complete the remainder of S_1 to a surface S_2 by placing over D_1 a cap T_1. This cap is constructed as in the proof of Theorem 1 but it is based on D_1. Also, let T_1 be such that $S_2\subset S_1$. Now repeat the same procedure on S_2, using $s_2=H_2\cap S_2$, β_2, v_2, D_2, P_2 and T_2 in place of s_1, β_1, v_1, D_1, P_1 and T_1. In this way one obtains a surface $S_3\subset S_2$. Let the process be continued. In the limit there results a surface S which will be shown to possess a dense set of p-universal points for an appropriate choice of the sequence $\{\beta_n\}$.

By construction, each T_n contains a p-universal point p_n. Therefore at the nth stage S_n there are $n-1$ such points $p_1, \cdots, p_{n-1}$. The sequence $\{\beta_n\}$ is to be selected so that the following conditions hold: (1) during the deforming of S_n into S_{n+1} the points $p_1, \cdots, p_{n-1}$, are left in place together with some neighbourhoods (these of course shrink with increasing n), and

(2) in the limit the points $\{p_n\}$, $n=1, 2, \cdots$, still function as p-universal points on S.

The first condition is easily met by making $\{\beta_n\}$ tend to zero fast enough. The second condition is less simple but it can also be satisfied in the same way.

At the nth stage S_n let P^1 be the unique supporting plane of S_n, and also of S, at p_1. Let P_{x1} be the plane parallel to P^1 on the origin-side of it, and whose distance from P^1 is x. Let $C_{nx1}=P_{x1}\cap S_n$, let $f_{n1}(x)$ be the length of C_{nx1}, let $F_{nx1}=C_{nx1}\cap(S-\bigcap_{j=1}^{n} S_j)$, let $g_{n1}(x)$ be the length of F_{nx1}, and let $h_{n1}(x)=g_{n1}(x)/f_{n1}(x)$. Roughly speaking,

$h_{n1}(x)$ is the percentage modification of a parallel section near p_1, due to the deformations of the first $n-1$ stages. The p-universal point p_1 on S_2 will also be p-universal on the limit surface S if

$$\lim_{x\to 0,\, n\to\infty} h_{n1}(x) = 0.$$

Now the deformation functions $h_{nk}(x)$ are formed for other p-universal points p_k, as they appear in the process. The condition (2) above will be satisfied by selecting $\{\beta_n\}$ so that $h_{nk}(x) \leqq x$. This completes the description of S and the proof.

If $\varnothing$ is the empty set then trivially $\varnothing \subset \mathfrak{C}_p(s) \subset \mathfrak{C}(s) \subset \mathfrak{U}$. The next theorem is concerned with the position of the two middle sets between the two extreme ones. It is shown that the most radical situation may occur.

THEOREM 3. *There exists a surface S and a point s on it, such that $\mathfrak{C}_p(s) = \varnothing$ and $\mathfrak{C}(s) = \mathfrak{U}$.*

Consider the basis $\{C_n\}$ used in the proof of Theorem 1. Choose q_n on C_n so that the radius of curvature of C_n attains its maximum γ_n at q_n; let also δ_n be its minimum. Let $OXYZ$ be a Cartesian coordinate frame and let P denote the OXY plane. Let $\{\lambda_n\}$ be a sequence of constants, decreasing steadily to 0 and with $\lambda_1 = 1$. Consider the sequence $\{\lambda_n C_n\}$ and place its members in P as follows: all the points q_n coincide with the origin O, $\lambda_n C_n$ lies on the positive x-side of the y-axis and is tangent to it at O, and $\lambda_{n+1} C_{n+1}$ is inside $\lambda_n C_n$. To satisfy the last condition let $0 < \lambda_n < \lambda_{n-1}\delta_n/\gamma_n$. Let P_n be a plane through the y-axis at an angle θ_n to P. The sequence $\{\theta_n\}$ is steadily decreasing to 0 and $\theta_1 = \pi/2$. Transfer $\lambda_n C_n$ from P to P_n by rotating it through an angle θ_n about the y-axis, and denote the result by $\lambda_n C_n'$. Now select $\{\theta_n\}$ and $\{\lambda_n\}$ so that a surface L can be constructed to satisfy these conditions: $L \cap P_n = \lambda_n C_n'$, L possesses at the origin the unique supporting plane P, and the intersection of L with the OXZ-plane is a curve whose radius of curvature at 0 is either vanishing or infinite. Intersect L with the OYZ-plane and let M be the cap based on OYZ and containing the curves $\lambda_n C_n'$. Complete M to a surface S as follows. The intersection of S and the plane $z = \alpha$, on the negative x-side, is an ellipse with axes $a(\alpha)$ and $b(\alpha)$. These ellipses are determined so that $a(\alpha)/b(\alpha)$ tends to 0 together with α, and P is the unique supporting plane of S at the origin. Now the surface S and the origin s satisfy the conditions of the theorem: s is universal by the construction of M, so that $\mathfrak{C}(s) = \mathfrak{U}$; on the other hand, the curvature conditions on S imply that the plane sections of S at s possess two radii of curvature one of which (in the OYZ-plane) is γ_1 while the other one (in the OXZ-plane) is either zero or infinite. This is easily seen to preclude the existence of any parallel limit section at s. Therefore $\mathfrak{C}_p(s) = \varnothing$.

4. The definitions of $\mathfrak{C}(s)$ and $\mathfrak{C}_p(s)$ suffer from the dependence of the limit sections on the sequences of intersecting planes. A point s of a surface S will be called ordinary if $\mathfrak{C}_p(s) = \{\lambda C\}$ for a fixed C in $\mathfrak{U}$, that is, if the parallel limit sections at s do not depend on the choice of the sequence of the intersecting planes. A surface is ordinary if all its points are ordinary.

THEOREM 4. *There exists an ordinary surface with a universal point.*

First it will be shown that for any C_n in the previously used basis $\{C_n\}$ there exists a surface of class C^∞ at all points except for one point s for which $\mathfrak{C}_p(s) = \{\lambda C_n\}$. Let K be a cone with the directrix C_n. Transform K into N by a compression in the axial direction. Let the compression be sufficiently strong near the vertex of K to send it into a point s on N, at which N possesses a unique supporting plane. Also, let N be of class C^∞ everywhere except at s. On N take now any cap containing s and complete it to a surface which is of the class C^∞ everywhere except at s.

Let S_1 be the unit sphere about the origin and let A be an arc of a great circle, terminating at the north pole s of S_1. On A let $\{s_n\}$ be a sequence of points tending steadily to s. About s_n as centre draw on S_1 a small circle D_n of radius ρ_n. The radii are such that no two circles intersect. Let V be the remainder of S_1 after the removal of the spherical caps based on all the D_n's. For the curve C_n of the previously used basis construct a cap V_n based on D_n and of class C^∞ everywhere except for one point v_n for which $\mathfrak{C}_p(v_n) = \{\lambda C_n\}$. As shown in the preceding paragraph, this can be done. Complete V to a surface S by placing V_n over D_n on V. Moreover, let S be of class C^∞ at all points except at the ordinary points v_n and at their limit s. Let $h(x)$ be the deformation function at s for the deformation of S_1 into S. It is defined in the same way as in the proof of Theorem 2. Now select the sequences $\{s_n\}$ and $\{\rho_n\}$ so that $h(x)$ tends to 0 together with x. The surface S is ordinary at all points: at the sequence $\{v_n\}$ by construction, at s by the condition on $h(x)$, and at all other points by the property of C^∞. On the other hand, it easy to show that s is a universal point.

5. Many other similar questions could be raised. Can the set $\mathfrak{C}(s)$

be empty? Does there exist an ordinary p-universal surface? What does the condition $\mathcal{C}(s)=\{\lambda C\}$, C fixed, imply? While for a surface of class C^2 a parallel limit section is an ellipse and for a surface of class C^3 any limit section is an ellipse, can a surface of class C^2 possess a nonelliptical limit section? What are the invariant properties, if any, of the set $\mathcal{C}(s)$ and $\mathcal{C}_p(s)$ under the change of s on the surface, and under the transformations of the surface itself?

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Bibliography

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