THE STRUCTURE OF A SPECIAL CLASS OF RINGS[1]

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A ring R such that for every $x \in R$ there exists an element $c(x) \in R$ (depending on x) such that $x^2c(x) - x$ is in the center will be called a ξ-ring [**9**].[2] It is our purpose in the present paper to study the following conjecture of I. N. Herstein: Every ξ-ring R is a subdirect sum of rings R_α where either (1) R_α is a division ring or (2) there exists an ideal P_α in the center of R_α such that $P_\alpha = R_\alpha$ or R_α/P_α is a field. The validity of this conjecture will be shown in the presence of certain additional restrictions on the ring R. In general we shall prove a similar theorem in which the possibility (2) is weakened so as to read, "Every commutator of R_α is in the center."

1. **The general solution.** Our ξ-ring will be denoted by R, the center by Z, the (Jacobson) radical by N, the set of nilpotent elements by I, and the commutator ideal by C.

THEOREM 1. *If R is a ξ-ring, then $xc(x) = c(x)x$ for all $x \in R$.*

PROOF.[3] For the sake of convenience we shall write c for $c(x)$. If $a \in R$ satisfies $a^2 = 0$, then since $a^2c(a) - a \in Z$ we have $a \in Z$. Let $x \in R$. Since $x^2c - x \in Z$, $(x^2c - x)x = x(x^2c - x)$ and so we obtain (1) $x^2(xc - cx) = 0$. Using (1) $[x(xc - cx)x]^2 = 0$ and so (2) $x(xc - cx)x \in Z$. Commuting this with x and using (1) we arrive at (3) $x(xc - cx)x^2 = 0$. Since $x^2c - x \in Z$ we can easily verify that (4) $xc - cx = [x(xc - cx) + (xc - cx)x]c$. We left multiply (4) by x; using (1) this simplifies to (5) $x(xc - cx) = x(xc - cx)xc = cx(xc - cx)x$ by (2). We right multiply (5) by x and so get (6) $x(xc - cx)x = cx(xc - cx)x^2 = 0$ by (3). Reapplying the result of (6) to (5), (5) reduces to (7) $x(xc - cx) = 0$. This result simplifies (4) to (8) $xc - cx = (xc - cx)xc$. From (7) it follows that $[(xc - cx)x]^2 = 0$ and so $(xc - cx)x \in Z$; com-

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[2] Utumi has shown in [9] that any ξ-ring modulo the maximal nil ideal is a subdirect sum of division rings and commutative rings. His arguments are along slightly different lines than those used in this paper.

[3] The proof given here is a direct one due to Dr. Herstein. We mention also that the concept of subdirect irreducibility, first exploited by Herstein in [**4**], is used strongly in the present paper.

muting this with x and using (7), we have (9) $(xc-cx)x^2=0$. Since $(xc-cx)x\in Z$ (8) becomes $(xc-cx)=c(xc-cx)x$, and so (10) $(xc-cx)x = c(xc-cx)x^2=0$ from (9). But then (8) reduces to $xc-cx=0$ which is the desired result.

LEMMA 1. *Let $x\in R$. Then there exist elements $c_n\in R$, $n=1, 2, \cdots$, such that $x^{2n}c_n-x\in Z$.*

PROOF. The lemma is valid for $n=1$ by setting $c_1=c(x)$. For $n=2$ we choose a $d\in R$ such that $(x^2c_1)^2d-x^2c_1\in Z$. Applying Theorem 1, we set $c_2=c_1^2d$ and see that $x^4c_2-x=x^4c_1^2d-x=(x^2c_1)^2d-x = [(x^2c_1)^2d-x^2c_1]+(x^2c_1-x)\in Z$. Continuing in this fashion, we see that an easy induction will establish the lemma.

As an important corollary we have

LEMMA 2. *I is a two-sided ideal of R contained in the center Z.*

The following lemma is true in any ring in which every nilpotent element is central.

LEMMA 3. *Every idempotent of R is in the center.*

THEOREM 2. *Every primitive ξ-ring R is a division ring.*

PROOF. Since T is primitive there exists a maximal right ideal J such that $[J:R]=(0)$, where $[J:R]=\{x\in R \mid Rx\subset J\}$.

Suppose $J\neq(0)$. A contradiction will result provided we produce in J a nonzero central element, for if $0\neq z\in J\cap Z$, then $Rz=zR\subset J$ and thus $[J:R]\neq(0)$. To this end we choose a nonzero element $a\in J$ and, setting $c=c(a)$, note that $z=a^2c-a\in J\cap Z$. Without loss of generality we may assume that $z=0$, in which case $(ac)^2-ac =(aca-a)c=(a^2c-a)c=zc=0$, using Theorem 1. By Lemma 3 the idempotent $ac\in J\cap Z$, and it is nonzero because otherwise $a^2c-a = -a\in Z$ and so $0\neq a\in J\cap Z$. We have completely disposed of the possibility that $J\neq(0)$.

When we consider the case that the maximal right ideal J is zero, it is well known that either R is a division ring or $R^2=(0)$, and the latter case is ruled out because $N=(0)$.

COROLLARY 1. *Every semi-simple ξ-ring R is a subdirect sum of division rings.*

COROLLARY 2. *Every subdirectly irreducible semi-simple ξ-ring is a division ring.*

Since any ring is a subdirect sum of subdirectly irreducible rings, the second corollary to Theorem 2 reduces our study of ξ-rings to that of subdirectly irreducible ξ-rings with nontrivial radical. R will now be assumed to be subdirectly irreducible with minimal two-sided ideal $S \neq (0)$ and $N \neq (0)$.

LEMMA 4. *$S^2 = (0)$ and $S \subset Z$.*

PROOF.[4] Let $0 \neq s \in S$ and choose a $c \in R$ such that $s^2c - s = z \in Z \cap S$. Since $S \subset N$ we know that $z \neq 0$, for otherwise s would be a radical multiple of itself. Either the two-sided ideal $zS = S$ or $zS = (0)$, by the definition of S. $zS = S$ is impossible, because z would be a radical multiple of itself. Thus $zS = (0)$. Now, choosing any $t \in S$, we see of course that $0 = zt = tz = ts^2c - ts$. In other words, $(ts)(sc) = ts$, forcing $ts = 0$, since ts is a radical multiple of itself. As s and t were completely arbitrary elements of S we have shown that $S^2 = (0)$. In particular $s^2 = 0$ for all $s \in S$, so by Lemma 2 we may conclude that $S \subset Z$.

In any ring it is easy to verify the following

LEMMA 5. *If U is any ideal of R contained in the center of R, then $CU = (0)$, where C is the commutator ideal of R.*

LEMMA 6. *If $xy \in S$ for some $x \in C$ and some $y \in R$, then $xy = yx$.*

PROOF. Lemma 5 implies that $(yx)^2 = y[(xy)x] = 0$ since $xy \in S \subset Z$ and $x \in C$. Thus yx as well as xy is contained in the ideal $I \subset Z$. We now choose a $c \in R$ such that $x^2c - x \in Z$ and, using Theorem 1, write $xy - yx = (x^2c)y - y(x^2c) = c[x(xy)] - [(yx)x]c$. The right hand side is zero since $x \in C$, $xy \in I$, and $yx \in I$. It follows that $xy = yx$.

LEMMA 7. *Let $a \in C$, y, $r \in R$, and $s \in S$ such that $s = (ay - ya)r$. Then $s = 0$.*

PROOF. We select an element $c \in R$ such that $a^2c - a \in Z$ and write $v = ay - ya = a^2cy - ya^2c = a^2(cy - yc) + a(ay - ya)c + (ay - ya)ac$. This means that $s = vr = a^2(cy - yc)r + avcr + vacr$. Noting that $S \subset Z$ and v, ac, $vac \in C$, we see by Lemma 5 and Lemma 6 that $0 = s(ac) = (vr)(ac) = (rv)(ac) = r(vac) = (vac)r$. A similar argument shows that $avcr = 0$. We are thus left with $0 \neq s = a^2(cy - yc)r$. Again, using Lemma 5 and Lemma 6, we have $0 = s(cy - yc)r = [a(a(cy - yc)r)](cy - yc)r = [(a(cy - yc)r)a](cy - yc)r = [a(cy - yc)r]^2$. By Lemma 2 $a(cy - yc)r \in I \subset Z$. $s = a[a(cy - yc)r] \in CI = (0)$ by Lemma 5, and the proof is complete.

THEOREM 3. *If R is a subdirectly irreducible ξ-ring with nontrivial radical, then every commutator of R is in the center.*

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[4] See Herstein [6, p. 107].

PROOF. Let $a=ef-fe$ be a commutator of R, and, if $y\in R$, consider the element $v=ay-ya$. We choose an element $d\in R$ such that $v^2d-v\in Z$ and form the two-sided ideal $(v^2d-v)R$. If $(v^2d-v)R\neq(0)$, then $S\subset(v^2d-v)R$, yielding $0\neq s=(v^2d-v)t=vr$ for suitable $s\in S$ and t, $r\in R$, a contradiction of Lemma 7. If $(v^2d-v)R=(0)$, then in particular $(v^2d-v)d=0$, and, making use of Theorem 1, vd is an idempotent. By Lemma 3, $vd\in Z$ and so vdR is a two-sided ideal. $vdR\neq(0)$ is again ruled out because of Lemma 7, and, recalling that $(v^2d-v)R=(0)$, we have $vR=(ay-ya)R=(0)$. Since $aee(fe)-(fe)e$ is also a commutator, we may replace a by ae in the preceding discussion and obtain $[(ae)y-y(ae)]R=(0)$. In particular $a^3=a(ef-fe)a$ $=(aef-fae)a+(fae-afe)a=[(ae)f-f(ae)]a+[(fa-af)e]a=0$. Using Lemma 2, we have finally $a\in Z$.

We now consider any arbitrary ξ-ring R and remark that it can be written as a subdirect sum of subdirectly irreducible ξ-rings R_α, since homomorphic images of ξ-rings are again ξ-rings. Applying Theorem 3 and Corollary 2 of Theorem 2 we have finally proved the

MAIN THEOREM. *Every ξ-ring R is a subdirect sum of subdirectly irreducible rings R_α where either R_α is a division ring or every commutator of R_α is in the center of R_α.*

We shall again confine our attention to subdirectly irreducible ξ-rings with nontrivial radical. Theorem 3 tells us that every commutator is central, and it is well known that in this situation $C^2=(0)$. Therefore we now have $(0)\neq S\subset C\subset I\subset N\cap Z$, provided R is not already commutative.

Let x, $y\in R$. We assert that $xy=yx+ayx$, where $a=a(x,y)\in C$. In fact, if c, $d\in R$ such that $x^2c-x\in Z$ and $y^2d-y\in Z$, we can write $xy-yx=x^2(cy-yc)+2xc(xy-yx)=x^2(yc-cy)(2yd)+x^2y^2(dc-cd)$ $+2xy(yx-xy)(2yd)+(2xc)y^2(dx-xd)=ayx$, making liberal use of $C\subset Z$. An immediate corollary to this result is that every right (left) ideal of R is two-sided. In case R has an identity element we have shown that $xy=\lambda yx$ for all x, $y\in R$, where $\lambda=\lambda(x,y)=1+a$ is an invertible central element, since $a\in N$.

Let $A(S)$ be the set $\{a\in R \mid aS=(0)\}$. We shall develop briefly some results already studied by Herstein [6, pp. 109–110]. Our first assertion is that $A(S)$ coincides with the union of left and right zero divisors of R. Indeed, let $a\neq 0\in R$ be a (left) zero divisor, i.e., $ab=0$ for some $b\neq 0$. If $T=\{x\in R \mid ax=0\}$, then $(0)\neq S\subset T$ since T is a nonzero right ideal; therefore $aS=(0)$.

We claim next that $NS=(0)$. Indeed, suppose that NS, being an ideal contained in S, is equal to S. Then for some nonzero element

$s \in S$ we have $Ns \neq (0)$ and hence $Ns = S$. A contradiction is then reached since s would be a radical multiple of itself. Thus $N \subset A(S)$ and we can extend our information about R to $(0) \neq S \subset C \subset I \subset Z \cap N \subset N \subset A(S)$.

We finally prove that either $R = A(S)$ or else $\bar{R} = R/A(S)$ is a field. Suppose that $R \neq A(S)$. $\bar{R}$ is commutative since $C \subset A(S)$. There exists an $s \neq 0 \in S$ such that $sR \neq (0)$ (because $R \neq A(S)$), and hence $S = sR$. Choosing an $e \in R$ such that $se = s$, we see that $ex - x \in A(S)$ for all $x \in R$, since $s(ex - x) = (se)x - sx = 0$ and all zero divisors are in $A(S)$. It follows that the element $\bar{e}$ serves as the identity in $\bar{R}$. Now suppose $x \notin A(S)$. Then $0 \neq xs = t \in S$, and, from $tR \neq (0)$, we can select a $y \in R$ such that $ty = s$. Thus $(xy)s = (xs)y = ty = s$ and $(xy)s = (xy)(se) = se$, or $(xy - e)s = 0$. This shows that $xy - e \in A(S)$ and that $\bar{y}$ is an inverse for $\bar{x}$. The assertion that $\bar{R}$ is a field or $A(S) = R$ has now been demonstrated.

We remark finally in this section that all our results for ξ-rings up to this point could just as well have been derived for ξ-algebras.

2. **Special cases.** Before beginning our study of certain special ξ-rings we state a simple but well known result in ring theory which will be of general use throughout the present section.

LEMMA 8. *Let R be either a ring or an algebra (over a field Φ), and suppose for some x, $y \in R$ that $xy - yx$ is central. Let $p(x) = \sum_{i=0}^{n} \alpha_i x^i$ be a polynomial in x, where the α_i are integers (if R is a ring) or the $\alpha_i \in \Phi$ (if R is an algebra). Denote by $p'(x) = \sum_{i=1}^{n} i\alpha_i x^{i-1}$ the formal derivative of $p(x)$. Then $p(x)y - yp(x) = p'(x)(xy - yx)$.*

We shall next study ξ-rings R under the restriction that for all $x \in R c(x)$ is in the subring (x, Z) generated by x and the center Z. Since the quaternions satisfy this property, our main hope of obtaining a sharper picture of R lies in considering the case where R is subdirectly irreducible with nontrivial radical.

THEOREM 4. *Let R be a ξ-ring in which $S \neq (0)$, $N \neq (0)$, and $c(x) \in (x, Z)$ for all $x \in R$. Then $A(S) \subset Z$ and either R is commutative or $R/A(S)$ is a field.*

PROOF. Suppose $x \in A(S)$. By assumption $c = c(x) = \sum_{i=0}^{n} z_i x^i$ for suitable $z_i \in Z$. Let $y \in R$. Then $xy - yx = (x^2c)y - y(x^2c) = (\sum_{i=0}^{n} z_i x^{i+2})y - y(\sum_{i=0}^{n} z_i x^{i+2}) = \sum_{i=0}^{n} z_i(x^{i+2}y - yx^{i+2}) = \sum_{i=0}^{n} (i+2)z_i x^{i+1}(xy - yx) = a(xy - yx)$ where $a \in A(S)$, using the fact that $xy - yx \in Z$ by Theorem 3 and applying Lemma 8 to $x^{i+2}y - yx^{i+2}$ for all i. If $(xy - yx)R = (0)$, then $xy - yx = 0$ by the equation $xy - yx = a(xy - yx)$. If $(xy - yx)R \neq (0)$ we can find an $r \in R$

such that $0 \neq s = (xy - yx)r$, and a contradiction arises if we note that $(xy-yx)r = a(xy-yx)r = as = 0$ since $a \in A(S)$. Hence $xy - yx = 0$ for all $x \in A(S)$ and all $y \in R$, that is, $A(S) \subset Z$. Clearly R is commutative if $R = A(S)$, and, if $R \neq A(S)$, we know already from the previous section that $R/A(S)$ is a field.

THEOREM 4A. *Let R be a ξ-ring in which $c(x) \in (x, Z)$ for all $x \in R$. Then R is a subdirect sum of subdirectly irreducible rings R_α where either* (1) *R_α is a division ring or* (2) *there exists an ideal P_α of R_α contained in the center of R_α such that $P_\alpha = R_\alpha$ or R_α / P_α is a field.*

PROOF. We first apply the Main Theorem and note that the condition $c(x)$ carries over to the homomorphic images R_α. The proof is then immediate if for each R_α we set $P_\alpha = A(S)$ and apply Theorem 4.

Theorem 4 generalizes a result of Herstein [**7**, p. 868, Theorem 11]; in his paper $c(x)$ is a polynomial in x with integer coefficients. Our next specialization occurs in

THEOREM 5. *Let R be an algebraic ξ-algebra over a field Φ, in which $S \neq (0)$ and $N \neq (0)$. Then $I = N = A(S) \subset Z$, and R is either commutative or $R/A(S)$ is a field.*

PROOF. If $A(S) \neq I$ there is an $a \in A(S)$ such that a is not nilpotent. Since R is algebraic we can write $\sum_{i=0}^{n} \alpha_i a^i = 0$ for some $n \geqq 1$, where $\alpha_n = 1$ and $\alpha_i \in \Phi$ for $i = 0, 1, \cdots, n$. α_0 must be zero, for otherwise, choosing some $s \neq 0 \in S$, we would have $0 = (\sum_{i=1}^{n} \alpha_i a^i)s = -\alpha_0 s$, a contradiction. Therefore we now have $\sum_{i=m}^{n} \alpha_i a^i = 0$, where $m \geqq 1$ and $\alpha_m \neq 0$. If $a^m R \neq (0)$, then $S \subset a^m R$, so we can choose an $r \in R$ such that $0 \neq s = a^m r \in S$. Then $0 = (\sum_{i=m}^{n} \alpha_i a^i) r = \alpha_m a^m r + \sum_{i=1}^{n-m} \alpha_{m+i} a^i (a^m r)$ $= \alpha_m s + \sum \alpha_i a^i s = \alpha_m s$ since $a \in A(S)$. This is an impossible situation since $\alpha_m \neq 0$ and $s \neq 0$. Thus $a^m R = (0)$, and in particular $0 = a^m a = a^{m+1}$ so a is nilpotent, contradicting our original assumption about a. Our argument shows that $A(S) \subset I$, and, since we already know that $I \subset A(S) \cap Z$, we can safely assert that $I = N = A(S) \subset Z$. The rest of the theorem follows easily.

Theorem 5 can be sharpened if we impose the condition that Φ is perfect.

THEOREM 6. *Let R be an algebraic ξ-algebra over a perfect field Φ in which $S \neq (0)$ and $N \neq (0)$. Then R is commutative.*

PROOF. Suppose there exists an $x \in R$ such that $x \notin Z$. By Theorem 4 $x \notin A(S)$, and $\overline{R} = R/A(S)$ is an algebraic extension field of Φ. Let $p(t) = \sum_{i=0}^{n} \alpha_i t^i$ be the minimal polynomial over Φ satisfied by $\bar{x} \in \overline{R}$. The degree of $p(t) > 1$, for otherwise $\alpha_0 + x \in A(S)$, forcing $x \in Z$.

Since Φ is perfect, $p(t)$ is a separable polynomial, which means that its derivative $p'(t)=\sum_{i=1}^{n} i\alpha_i t^{i-1}$ is not identically zero. Now let $y\in R$ such that $xy-yx\neq 0$. We now use the fact that $p(x)\in Z$ and $xy-yx\in Z$ (Theorem 3), together with an application of Lemma 8, to write the equation $0 = p(x)y - yp(x) = (\sum\alpha_i x^i)y - y(\sum\alpha_i x^i) = \sum\alpha_i(x^i y - yx^i) = (xy-yx)(\sum i\alpha_i x^{i-1}) = (xy-yx)p'(x)$. It follows that $p'(x)\in A(S)$ because $xy-yx\neq 0$. In other words, $p'(\bar{x})=0$ in $\bar{R}$, contrary to the minimality of $p(t)$. Therefore we must assume that $R=Z$, and the proof is complete.

Both forms of the Main Theorem corresponding to Theorem 5 and to Theorem 6 can now be sharpened in a natural way.

THEOREM 5A. *If R is an algebraic ξ-algebra, then R is a subdirect sum of subdirectly irreducible algebras R_α where either* (1) *R_α is a division algebra or* (2) *there exists an ideal P_α of R_α contained in the center of R_α such that either R_α is commutative or R_α/P_α is a field.*

THEOREM 6A. *If R is an algebraic ξ-algebra over a perfect field, then R is a subdirect sum of subdirectly irreducible algebras R_α which are either division algebras or commutative algebras.*

We reproduce an example due to McLaughlin and Rosenberg [**8**, pp. 207–208] to show that the restriction that Φ is perfect is necessary in Theorem 6. At the same time, of course, we are furnished with a nontrivial illustration of Theorem 3 and Theorem 5.

Let Φ be the two element field and $H=\Phi(x, y)$ the field of rational functions in two indeterminates x and y over Φ. J is the algebra $H+Ha$, where multiplication is defined by:

$$(g_1 + h_1 a)(g_2 + h_2 a) = (g_1 g_2) + (h_1 g_2 + g_1 h_2)a \text{ for } g_1, g_2, h_1, h_2 \in H.$$

Finally R is the algebra $J+Ju+Jv+Jw$, with the multiplication table:

	u	v	w
u	x	w	xv
v	$a+w$	y	$yu+av$
w	$au+xv$	yu	$xy+aw$

It is straightforward to check that R is actually an associative algebra (of finite dimension) over H. It can be verified that R is subdirectly irreducible with (using our usual notation) $(0)\neq S=aR=I=N$

$=A(S)\subset Z$. $R/A(S)$ is a field and R is a ξ-algebra. We point out, finally, that R is not commutative since, e.g., $uv-vu=-a\neq 0$.

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