LEMMA 2′. *Let L be a distributive lattice with* 0 *element and with finite bounded chains.* $L_\cup$ *may be imbedded in the discrete cardinal product of as many copies of L as there exist join-irreducible elements in L.*

THEOREM 3′. *Let L be a lattice with* 0 *element and with finite bounded chains. All join-endomorphisms of L form a distributive lattice if and only if L is distributive.*

BUDAPEST, HUNGARY

LINEAR COMPLETENESS AND HYPERBOLIC TRIGONOMETRY

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In this paper we show the uniqueness of the relation between a segment and its angle of parallelism as derived from a model. Upon generalizing this relation hyperbolic trigonometry follows in a remarkably simple way.

To introduce proper terminology [**5**, pp. 11–28] let Σ denote an *axiom system*, that is a certain set of axioms together with the undefined technical and logical or universal terms used to state the axioms. We define the terms *interpretation* and *model* in the usual fashion. It is useful to make a clear distinction between the following three concepts. (1) A *Σ-statement* is a meaningful expression, not necessarily true, in the technical and universal terms of Σ. (2) A *T-Σ-statement* is a true Σ-statement in the sense of being logically derivable from Σ. (3) If I denotes an interpretation of Σ, then an *I-Σ-statement* is a Σ-statement holding for the model which results from the interpretation I.

For the purpose of this paper let Σ be the postulate system of Hilbert [**3**, pp. 2–30] with the Euclidean axiom of parallelism replaced by the characteristic postulate of hyperbolic plane geometry [**6**, p. 66]. Some authors have used models to find I-Σ-statements [**1**, §39–117; **2**]. Such a procedure, however, may be objectionable [**1**, §118]. Conceivably an I-Σ-statement could be made which is not a T-Σ-statement, but is merely a property of a particular model. In other words, it might be possible to find contradictory I-Σ-statements in two different models. Clearly, if this happens it indicates that our system Σ is not complete [**5**, pp. 33–36]. Any I-Σ-statement that is not a T-Σ-statement would still be compatible with the axioms of Σ.

Presented to the Society April 19, 1958; received by the editors February 28, 1958.

The special Σ-statement to be considered here is the relation between a distance and its corresponding angle of parallelism [**4**, pp. 143–144]. To avoid ambiguity we assign the unit of length to a segment whose angle of parallelism is 2 arc tan e^{-1}. Assume now that for a prescribed segment, not of length one, two different angles of parallelism are found in two models. This would be the situation, suggested above, of two contradictory I-Σ-statements. We propose to show that two different formulas for the angle of parallelism are impossible in a geometry based on Σ. To this end, let x denote the given segment perpendicular to a line MN, θ the smaller, and θ' the larger of the angles of parallelism. If θ' were the true angle, lines passing through the proper end point of x and making angles greater than or equal to θ and less than θ' with x, would intersect MN [**6**, p. 67]. The line MN would then have more points than in the case of θ being the angle of parallelism. This contradicts the postulate of linear completeness [**3**, p. 30]. Hence the conflicting I-Σ-statements on the angle of parallelism cannot both be compatible with Σ. The functional relationship between x and θ must be unique and the same in all models. It is consistent with the axioms of Σ; whether it is a T-Σ-statement, that is provable from these axioms without additional assumptions is not of concern to us here.

The relationship in question derived from two different, though closely related, models [**1**, §74; **6**, pp. 214–216] is $e^{-x}=\tan \theta/2$. We admit obtuse angles for x negative [**6**, p. 77] for the purpose of the present paragraph. In order to generalize this formula we consider two parallel lines which intersect a third line in two points P, Q such that distance $PQ=z$. Let θ and ϕ be the oblique angles in the triangle determined by P, Q, and the ideal point of the given parallels [**6**, pp. 71–75]. The distances corresponding to these angles, regarded as angles of parallelism, are denoted by x and y. There is exactly one line, perpendicular to PQ and parallel to the first one of the given parallel lines. Its distance from P is equal to x. Because of the transitivity of parallelism [**4**, p. 139] this line perpendicular to PQ is also parallel to the second one of the given parallels and its distance from Q is necessarily y. Since $z=x+y$, we conclude that

$$e^{-z} = \tan \theta/2 \tan \phi/2. \tag{1}$$

In (1) z is positive for the angles chosen. Relation (1) is readily changed to

$$\cosh z = \frac{1 + \cos\theta \cos\phi}{\sin\theta \sin\phi} \cdot \tag{2}$$

At this point let the customary notation apply to a right triangle ABC. Through the vertex B we draw the two parallels to line AC and designate by θ the acute angle of parallelism corresponding to side a. Using (2) with respect to the hypotenuse c and the parallels in either sense we obtain

$$\cosh c = \frac{1 + \cos \alpha \cos (\theta + \beta)}{\sin \alpha \sin (\theta + \beta)}, \tag{3}$$

$$\cosh c = \frac{1 - \cos \alpha \cos (\theta - \beta)}{\sin \alpha \sin (\theta - \beta)} \cdot \tag{4}$$

Equating (3) and (4) we have $\cos \alpha \sin \theta = \sin \beta$. By means of a suitable formula for the angle of parallelism [6, p. 151] this takes the form

$$\cos \alpha \operatorname{sech} a = \sin \beta. \tag{5}$$

Hence by analogy,

$$\cos \beta \operatorname{sech} b = \sin \alpha. \tag{6}$$

We now expand (3), use formulas for θ, and apply (5). Thus,

$$\cosh c = \cosh a \cosh b. \tag{7}$$

It is easily seen that (5), (6), (7) allow us to derive the remaining formulas for the right triangle.

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