A SUFFICIENT CONDITION THAT A MONOTONE IMAGE OF THE THREE-SPHERE BE A TOPOLOGICAL THREE-SPHERE

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1. A continuous transformation of one space onto another is called monotone provided the complete inverse set for each point of the image space is connected. A monotone image of a circle is a simple closed curve or a point. A monotone image of a 2-sphere is a configuration known as a cactoid, i.e. a peano space in which every true cyclic element is a topological 2-sphere. R. L. Moore has shown that if a monotone transformation of a 2-sphere has the additional property that no inverse set separates the 2-sphere, then the image space is again a topological 2-sphere or a point [3]. In the case of the three-sphere, S^3, as one would expect, the situation is more complicated and extra conditions need to be imposed if the image space is to be expected to look like an S^3.

A recent example of R. H. Bing [1] shows that if a monotone transformation on S^3 has the property that for each point of the image the complement of the inverse image is an open 3-cell, the image may not be a topological S^3, thus answering a long standing conjecture. By studying this example and profiting by conversations with Professor Bing the author was led to the following theorem.

2. THEOREM 1. *Let $M=f(S^3)$, where f is a monotone, continuous map such that* (i) *if $Y=\{y\in M \mid f^{-1}(y)$ does not reduce to a point$\}$, then given $y\in\overline{Y}$, and $\epsilon>0$, there is a topological 2-sphere K in $S(y, \epsilon)$ separating y and $M\backslash S(y, \epsilon)$ such that K does not meet $\overline{Y}$. Then M is a topological 3-sphere.*[2]

PROOF. Let $\epsilon_1>\epsilon_2>\cdots\to 0$ and $\sum\epsilon_i<+\infty$. The set $\overline{Y}$ is totally disconnected. Hence $\overline{Y}=Y_1\cup\cdots\cup Y_{n_1}$ where Y_i is closed,[3] $Y_i\cap Y_j=\square$ and $\delta(Y_i)<\epsilon_1/4$. Suppose $\eta_1=\min\rho[Y_i, Y_j]$, $i\neq j$. De-

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[2] $\overline{Y}$ = closure of Y.

[3] $\delta(Y)$ represents the diameter of Y, $S(y,\epsilon)$ is the set of points each of whose distance from y is less than ϵ. The symbol ρ represents the metric of the space concerned. It will be clear whether ρ refers to M or S^3 by noting in which space the sets are given.

If K is a topological 2-sphere in $M\backslash\overline{Y}$, the complement of $\overline{Y}$ in M, Int K $=f[\text{Int} f^{-1}(K)]$.

fine $\epsilon_1' = \min(\epsilon_1/4, \eta_1/3)$. A finite number of topological 2-spheres $K_1', \cdots, K_{m_1}'$ are found, by use of (i), such that for $i=1, \cdots, m_1$,

(1) $$\delta(K_i') < \epsilon_1';$$

(2) $$\bigcup \operatorname{Int} K_i' \supset \overline{Y};$$

(3) $$K_i' \cap \overline{Y} = \square.$$

The first set of operations is designed to replace the spheres $K_1', \cdots, K_{m_1}'$ by a set $\tilde{K}_1', \cdots, \tilde{K}_{p_1}'$ that enjoy properties similar to (1), (2), (3) and the further requirement

(4) $$\tilde{K}_i' \cap \tilde{K}_j' = \square, \qquad i \neq j.$$

The set of spheres $\tilde{K}_1', \cdots, \tilde{K}_{p_1}'$ may be found as follows. Since f^{-1} is topological on K_i', $L_i = f^{-1}(K_i')$ is a topological 2-sphere. Since $\rho[L_i, f^{-1}\overline{Y}] > 0$, we may apply the Bing approximation theorem [2] to find a polyhedral 2-sphere P_i as near L_i as we please so that P_i contains in its interior precisely those components of $f^{-1}\overline{Y}$ that are interior to L_i. By doing this for each i, we obtain a set of polyhedral 2-spheres

$$P_1, \cdots, P_{n_1}.$$

It may be supposed further that $P_i \cap P_j$ is a finite collection (possibly null) of pairwise disjoint simple closed curves, none of which may be removed by an arbitrarily small deformation of P_i or P_j. In addition,

(2′) $$\bigcup \operatorname{Int} P_i \supset f^{-1}(\overline{Y}).$$

(3′) $$P_i \cap f^{-1}(\overline{Y}) = \square.$$

We first describe how to find a set of polyhedral 2-spheres $\tilde{P}_1, \cdots, \tilde{P}_{p_1}$ such that conditions (2′), (3′) and the following hold

(4′) $$\tilde{P}_i \cap \tilde{P}_j = \square.$$

Suppose $C_1, \cdots, C_q$ are the components of $P_1 \cap P_2$. If $q=1$, let C_1 divide P_1 into U_1, V_1 and C_1 divide P_2 into U_2, V_2. Then P_1 and the closure of the component (V_2 say) of $P_2 \backslash C_1$ in the exterior of P_1 together with the appropriate disk (U_1 or V_1) gives a pair of 2-spheres P_1, P_2' that covers the same part of $f^{-1}(\overline{Y})$ that $P_1 \cup P_2$ does, neither P_1 nor P_2' meets $f^{-1}(\overline{Y})$ and, by a slight deformation $P_1 \cap P_2' = \square$.

If $q > 1$, at least one of $C_1, \cdots, C_q$, say C_1, will not separate $C_2, \cdots, C_q$ on P_1. (Of course C_1 may separate $C_2, \cdots, C_q$ on P_2, but that is irrelevant.) By replacing P_2 by 2 new polyhedral 2-spheres meeting along a disk on P_1, we again have covered the same part of

$f^{-1}(\overline{Y})$ and by a pair of slight deformations obtain 3 polyhedral 2-spheres

$$P_1,\ P_2',\ P_2''$$

such that the number of components of $P_1 \cap P_2'$ or $P_1 \cap P_2''$ is less than q.

Continuing, we obtain, after a finite number of such operations a collection of polyhedral 2-spheres

$$\tilde{P}_1, \cdots, \tilde{P}_{p_1}$$

such that

(2″) $$\bigcup \operatorname{Int} \tilde{P}_i \supset f^{-1}(\overline{Y}).$$

(3″) $$\tilde{P}_i \cap f^{-1}(\overline{Y}) = \square.$$

(4″) $$\tilde{P}_i \cap \tilde{P}_j = \square,\ i \neq j.$$

Define $\tilde{K}_i = f(\tilde{P}_i)$. We note that under the steps made in forming $\tilde{P}_i$, or, correspondingly, $\tilde{K}_i$, that the diameters of the spheres replacing K_j may be greater than that of K_j. However, since $\epsilon_1' < \eta_1(1/3)$, the definition of η_1 and the triangle inequality show that $\delta(\tilde{K}_i) < 3\epsilon_1/4 < \epsilon_1$. Hence $\tilde{K}_1, \cdots, \tilde{K}_{p_1}$ satisfy

(1) $$\delta(K_i) < \epsilon_1:$$

(2) $$\bigcup \operatorname{Int} K_i' \supset \overline{Y}:$$

(3) $$\tilde{K}_i' \cap \overline{Y} = \square,$$

(4) $$\tilde{K}_i' \cap \tilde{K}_j' = \square,\ i \neq j.$$

To $\epsilon_2 > 0$, write $Y_i = Y_{i,1} \cup \cdots \cup Y_{i,n_2}$, where $Y_{i,j}$ is closed, $Y_{i,j} \cap Y_{i,j'} = \square$ and $\delta(Y_{i,j}) < \epsilon_2/4$. Put

$$\eta_2 = \min \rho[Y_{i,j},\ Y_{i',j'}],\ \rho\left[Y_{i,j},\ \bigcup_1^{p_1} \tilde{K}_i\right].$$

Let $\epsilon_2' < \epsilon_2/4,\ \eta_2/3$. By the hypotheses (i) there is a finite collection of topological 2-spheres in M, $K_1^2, \cdots, K_{m_2}^2$ such that

(1) $$\delta(K_i^2) < \epsilon_2':$$

(2) $$\bigcup_1^{m_2} \operatorname{Int} K_i^2 \supset \overline{Y}:$$

(3) $$K_i^2 \cap \overline{Y} = \square.$$

By the choice of ϵ_2', $K_i^2 \cap \tilde{K}_j^1 = \square$. By modifications of the $K_1^2, \cdots, K_{m_2}^2$ precisely as above at the first stage we arrive at another set of spheres $\tilde{K}_1^2, \cdots, \tilde{K}_{p_2}^2$ such that

(1) $$\delta(\tilde{K}_i^2) < \epsilon_2,$$

(2) $$\bigcup_1^{p_2} \operatorname{Int} K_i^2 \supset \overline{Y},$$

(3) $$K_i^2 \cap \overline{Y} = \square,$$

(4) $$K_i^2 \cap K_j^2 = \square, \quad i \neq j$$

(5) $$K_i^1 \cap K_j^2 = \square.$$

The general step is now clear.

To $\epsilon_n > 0$ we find a finite set of topological 2-spheres

$$\tilde{K}_1^n, \cdots, \tilde{K}_{p_n}^n$$

such that

(1) $$\delta(\tilde{K}_i^n) < \epsilon_n,$$

(2) $$\bigcup_1^{p_n} \operatorname{Int} \tilde{K}_i^n \supset \overline{Y},$$

(3) $$\tilde{K}_i^n \cap \overline{Y} = \square,$$

(4) $$\tilde{K}_i^n \cap \tilde{K}_j^n = \square, \quad i \neq j$$

(5) $$\tilde{K}_i^n \cap \tilde{K}_j^p = \square, \quad p < n, \text{ all } i, j.$$

3. Let $F_1', \cdots, F_{p_1}'$ be p_1 disjoint cubes (topological 2-spheres) with centers on the x-axis and faces parallel to the co-ordinate planes. We take the cubes congruent to one another for convenience. Let $F_1^2, \cdots, F_{p_2}^2$ be a similar set of cubes of smaller size so that

$$F_i^2 \subset \operatorname{Int} F_j'$$

if and only if

$$K_i^2 \subset \operatorname{Int} K_j'.$$

Continuing, for each n we have

$$F_1^n, \cdots, F_{pn}^n$$

a collection of pairwise disjoint cubes so that

$$F_i^n \subset \operatorname{Int} F_j^{n-1}$$

if and only if

$$K_i^n \subset \operatorname{Int} K_j^{n-1}.$$

Without loss we may require that $\delta(F_i^n) < 1/n$.

The following lemma is stated without proof.

LEMMA. *If $Q_1, \cdots, Q_n$ are disjoint polyhedral 2-spheres in S^3, no one interior to any other, and if Q_0 is a large polyhedral cube containing $Q_1, \cdots, Q_n$ in its interior, the closed domain bounded by $Q_0, Q_1, \cdots, Q_n$ is tame. Further, any two domains so formed in this way are homeomorphic.*

4. Let P^0 be a large cube in S^3 containing $\tilde{P}_1, \cdots, \tilde{P}_{p_1}$ in its interior. Then $K^0=f(P^0)$ is a 2-sphere in M containing $\tilde{K}_1', \cdots, \tilde{K}_{p_1}'$ in its interior. Let M_0 be the region in M exterior to K^0. Then $\overline{M}_0$ is homeomorphic to $S^3\backslash\text{Int}\, P^0$ under f^{-1}. Hence there is a homeomorphism h_0 from $\overline{M}_0$ to $S^3\backslash\text{Int}\, F_0$. Let M_1=region in M bounded by $K^0\cup\bigcup_1^{p_1}\tilde{K}_i'$. Then, by the lemma, $\overline{M}_1$ is homeomorphic to the region in S^3 bounded by $F^0\cup\bigcup_1^{p_1}F_i'$. Let h_1 be a homeomorphic extension of h_0 from $\overline{M}_0$ to $\overline{M}_0\cup\overline{M}_1$. The next step is similar, except that M_2 is a union of a finite number of regions bounded by the sets

$$\bigcup_1^{p_1}\tilde{K}_i' \cup \bigcup_1^{p_2}\tilde{K}_i^2.$$

However, these regions are in 1-1 correspondence with the number of regions bounded by

$$\bigcup_1^{p_1}F_i' \cup \bigcup_1^{p_2}F_i^2,$$

hence, by the lemma, the extension of h_1 from $\overline{M}_0\cup\overline{M}_1$ to $\overline{M}_0\cup\overline{M}_1\cup\overline{M}_2$ can be carried out.

Continuing, a sequence of homeomorphisms $h_0, h_1, h_2, \cdots$ is defined so that each is an extension of the preceding and

$$h(x) = h_n(x)$$

maps $M\backslash\overline{Y}$ homeomorphically onto the complement of a Cantor set X in S^3.

Since nested sequences of connected sets in $M\backslash\overline{Y}$ correspond to nested sequences of connected sets in $S^3\backslash X$, it is easy to see that h and h^{-1} are both uniformly continuous, hence the extension $\tilde{h}$ of h carries M homeomorphically onto S^3.

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