NOTE ON NIELSEN TRANSFORMATIONS[1]

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Let $a_1, \cdots, a_n$ be free generators of the free group F_n, $c_1, \cdots, c_n$ those of the free group F_n', N an isomorphism of F_n onto F_n', and $b_i = N(a_i)$ the maps of the a_i under N. Then the b_i are generators not only of F_n' but of any quotient group G'. I shall say that the generators b_i of G' arise from the a_i by a Nielsen transformation, N. In case $F_n = F_n'$, N is (Nielsen) automorphism of F_n but need not be an automorphism of $G = G'$; nor is every automorphism of G expressible in terms of some N acting on a given set of generators. However, a given automorphism of G can be written as a Nielsen transformation acting on a suitably large set of generators (Theorem 2). I shall discuss this and related matters in the present paper.

The following definitions and notation will be used.

$G = \{a_1, \cdots, a_n; R_1, \cdots, R_k\} = \{a; R\}$ will denote the group given on n generators a_i and k generating relations

$$R_j(R_j = R_j(a_1, \cdots, a_n) = 1.$$

G' will denote the commutator subgroup of G.

$\bar{x}$ will denote the inverse of x.

$[x, y] = xy\bar{x}\bar{y}$.

$I \sim N$ will denote the following. If $G = \{a; R\} \simeq IG = \{b; S\}$ under the mapping $I(a_i) = v_i(b)$ and there exist relations $s_i'(b)$ in IG for which the elements $v_i(b)s_i'(b)$ generate b freely, then the isomorphism I' given by $I'(a_i) = v_i(b)s_i'(b)$ is a Nielsen transformation, N. I shall say that I is equivalent to N: $I \sim N$. Similarly for automorphisms, A, that is in case $IG = G$.

The group $K = \{a; a^p\}$ has, for $a = a_1$ and prime p, automorphisms A, given by $A(a) = v(a)$, for which $v(a)$ does not generate $F_1(a)$. For example, if $p = 3$, $A(a) = a^2$ is such an automorphism. However, $a^2 = \bar{a}$ in K, so that the automorphism given by $A'(a) = \bar{a} = a^2\bar{a}^3 = aR'$, a Nielsen transformation, of K, is equivalent to A. For $p > 3$ and prime $q < p$, $A(a) = a^q$ is an automorphism of K, but $A \nsim N$.

There exist other groups with one defining relation having non-Nielsen automorphisms:

THEOREM 1. *The group* $L = \{u, v; uuv\bar{u}\bar{v}\bar{v}\bar{u}vu\}$ *has an automorphism*

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A, $A(u)=\bar{v}\bar{u}vu$, $A(v)=\bar{u}\bar{v}\bar{u}A(\bar{u})$, which is not equivalent to a Nielsen transformation.

L is the group of Listing's knot, the automorphisms of which were first studied by Dehn [6].

PROOF. Suppose $A\sim N$. The commutator subgroup L' contains u (since the generating relation R implies $\bar{u}=[uv,\ \bar{u}\bar{v}]$), hence it contains all conjugates $v^k u\bar{v}^k$; moreover, L' is a free group generated by u and $vu\bar{v}=y$ [1]. The factor group $L/L'=F(v)$ is also given by L/C, where C is the normal subgroup generated by u, and A induces an automorphism of $L/C=\{u,\ v;\ u\}$. Since every Nielsen transformation of a group $G=\{a;\ R_1\}$ maps the defining relation R_1 onto a conjugate of the defining relation, or of its inverse, of the image group $K=\{b;\ S_1\}$ [1], the assumption $A\sim N$ implies that an automorphism N, equivalent to A, of L is given by $N(u)=xu^\epsilon\bar{x}$, $x\subset L$, $\epsilon=\pm 1$, hence that $A(u)=\bar{v}\bar{u}vu=xu^\epsilon\bar{x}$ in L, and $A^2(u)=\bar{u}=A(x)A(u^\epsilon)A(\bar{x})$ is a conjugate $wu\bar{w}$ of u for some element w in L. This implies $wu\bar{w}u=1$ in L, which will prove false.

If $w=1$ in L, $\bar{u}=u$ follows, which does not hold in the free group $L'(u,\ y)$; since $wu\bar{w}u$ is not the empty word, w is not in L' (otherwise $L'(u,\ y)$ would contain a nontrivial relation: $wu\bar{w}=f(u,\ y)$ $=g(u,\ y)u\bar{g}(u,\ y)$ would equal $\bar{u}$).

Let $f=f(u,\ y)$, $g=g(u,\ y)$ be elements in L'. Since every element of L has a unique representation of the form $v^h f$, $w=v^k f$ with $k\neq 0$. This gives $wu\bar{w}=v^k fu\bar{f}\bar{v}^k$; setting $fu\bar{f}=g$, $wu\bar{w}=v^k g\bar{v}^k$. If now $wu\bar{w}u$ is a relation in L, then $v^k g\bar{v}^k=\bar{u}$, and $g(u,\ y)=\bar{v}^k\bar{u}v^k$. Again, since u and y generate L' freely, no relation of L is a word in u and y alone, and so for any element a in L', $a=h(u,\ y)$ uniquely; the above stated equality implies that for some integer $k\neq 0$ a conjugate of u by v^k is the u, y-word $\bar{g}=f\bar{u}\bar{f}$.

To see that no such k exists, observe that L is extension of $F_2(u,\ y)$ by $F_1(v)$, given by

$$L=\{v,u,y;\ uuv\bar{u}\bar{v}\bar{v}\bar{u}vu,\ \bar{y}vu\bar{v}\}=\{v,\ u,\ y;\ vu\bar{v}=y,\ vy\bar{v}=\bar{u}y^3\}$$
$$=\{v,\ u,\ y;\ B(u)=y,\ B(y)=\bar{u}y^3\}$$

where B is the automorphism of $F_2(u,\ y)$ induced by conjugation with v. Thus, $v^k u\bar{v}^k=B^k(u)$, $\bar{v}^k uv^k=B^{-k}(u)$.

The automorphism B is characterized [5] by the matrix

$$M=\begin{pmatrix} 0 & 1\\ -1 & 3\end{pmatrix}$$

no power of which has eigenvalue -1. It follows that $v^k u\bar{v}^k$ cannot be equal to $f\bar{u}\bar{f}$ for any k. Thus, A is not equivalent to a Nielsen transformation. With this the theorem is proved.

Though the automorphism A, given by $A(v)=\bar{u}\bar{v}\bar{u}A(\bar{u})$, $A(u)=\bar{v}\bar{u}vu$, is not equivalent to a Nielsen transformation in $L=\{v, u; R\}$, L can be given various presentations in which $A=N$. An example with 5 generators is given in [2]. The fewest possible number of generators to give $A=N$ is 3; the presentation mentioned in the proof above gives $A(u)=y\bar{u}^2$, $A(y)=\bar{u}y\bar{u}^2y\bar{u}^2$, $A(v)=\bar{v}\bar{y}u\bar{y}$.

These two examples are specific to the group L; the following one leads to a general procedure.

$$L = \{u, v, x, y; \bar{u} = [uv, \bar{u}\bar{v}], x = \bar{v}\bar{u}vu, y = \bar{u}\bar{v}\bar{u}\bar{x}\}$$

yields $A(u)=x$, $A(x)=\bar{u}$, $A(v)=y$, $A(y)=u^2vu$, with $A=N$. In general, given any group G and an automorphism A of G, introducing $A^k(g)$ as new generators $g^*(k)$ for every power k of A (in case A has finite order, h, $|k|<h>0$) and every generator g, and adding the new relations $g^*(k)A^k(\bar{g})$, renders $A=N$. The following method is always finite for finitely generated G.

THEOREM 2. *Let $G=\{g; R\}$, A an automorphism of G, $A(g)=v(g)$; define $w(g)$ by $A(w(g))=g$; set $H=\{g, h; R, \bar{h}w(g)\}$, $A'(g)=h\bar{w}(g)v(g)$, $A'(h)=g$. Then $H=G$, $A'\sim A$, and $A'=N$.*

In other words, every automorphism of a group can be effected by a Nielsen transformation acting on double the number of generators.

PROOF. By the theorem of Tietze [3], $H=G$. Since $A'(g)=v(g)$ mod relations in H, $A'(w(g))=A'(h)=g$ mod relations in H, $A'\sim A$. Since g is an image under A', all g-words are in the image group; then, because $A'(g)=h\bar{w}(g)v(g)$, all h-words are in the image group; hence $A'=N$.

The same can be accomplished for a given isomorphism I of two groups.

THEOREM 3. *Let $G=\{g; R\}\simeq IG=H=\{h; S\}$, $I(g)=v(h)$. Define $w(g)$ by $I(w(g))=w(v(h))=h$, and set $G^*=\{g, g^*; R, g^*\bar{w}(g)\}$, $H^*=\{h, h^*; S, h^*\bar{v}(h)\}$, $I^*(g)=h^*$, $I^*(g^*)=h$. Then $G^*=G$, $H^*=H$, $I^*\sim I$, and $I^*=N$.*

In the case of the group L, there is a presentation for which: 1. every automorphism, as well as 2. every isomorphism (to a group on the same number of relations) is a Nielsen transformation, namely $\{v, u, y; R, \bar{y}vu\bar{v}\}$ above. 1. The automorphism group of L, modulo inner automorphisms, is generated by A given above, and P given by $P(v)=\bar{v}$, $P(u)=\bar{u}$ [2]. In this presentation, $A=N_1$ and as $P(y)$

$=P(vu\bar{v})=\bar{v}\bar{u}v=y\bar{u}^3$ in L, $P=N_2$. 2. If $IL=H=\{g, a, b; S_1, S_2\}$, then $H/H'=F_1$ and (up to a Nielsen transformation on H) $H/H'=F_1(g)$, so that $H'=F_2(a, b)$; hence $I(u)$ and $I(y)$ are automorphic images $A(a)$ and $A(b)$ of a and b in $F_2(a, b)$. Since $H/H'=F_1(g)=I(L/L')$ $=I(F_1(v))$, the image $I(v)$ must be of the form $g^\epsilon f(a, b)$ modulo relations of H, where $\epsilon=\pm 1$. The mapping $I(v)=g^\epsilon f(a, b)$, $I(u)=A(a)$, $I(y)=A(b)$ is clearly a Nielsen transformation.

The rest of the discussion concerns extensions of the free group on two generators, F_2, by the free cyclic group, F_1, in case the commutator subgroup of the extension is isomorphic to F_2.

THEOREM 4. *If* $G=\{g, a; S\}\simeq H=\{x, y; R\}$, $S=S_1$, $R=R_1$ *with* $H/H'\simeq F_1$, $H'\simeq F_2$, *then* G *is Nielsen image of one of three nonisomorphic groups.*

The proof is based on two lemmas; the verification of the first one is due to Magnus.[2]

LEMMA 1. *Set* $a_i=g^i a\bar{g}^i$, $i: 0, \pm 1, \cdots$; $\epsilon=\pm 1$, r, k *integers. The relation* S *of Theorem* 4 *is* (*a conjugate of* $S^{*\pm 1}=$) $a_1^r\bar{a}_2a_1^k a_0^\epsilon$.

PROOF. By considering the factor group G/G', it can be verified that for some Nielsen transform NG, $NG'\supset a$; thus $a\subset G'$ may be assumed. Then $G/G'=F_1(g)$, G' is generated by the a_i, and the sum of exponents of g in S is zero. A conjugate $g^hS\bar{g}^h=f(a_0, \cdots, a_{r+h=L})$ generates the normal subgroup $\{S\}\subset F_2(g, a)$, so that one may assume $S=S(a_0, \cdots, a_L)$, with a_0 and a_L actually occurring in S.

Define K_t as that subgroup of G' generated by the elements $a_0, a_1, \cdots, a_L, \cdots, a_t$. Then K_L is a free subgroup of G' having $L+1$ generators and the one relation $S(a_0, \cdots, a_L)=S_L$. Since S_L is not the empty word, so that K_L is not isomorphic to the free group on $L+1$ generators, its rank is at most L. Since the Betti number of K_L abelianized is L, the rank is exactly L. Similarly for every K_t; thus $K_t\simeq F_L$. This shows: the ascending chain

$$K_L \subseteq K_{L+1} \subseteq \cdots$$

consists of subgroups of rank L of G', with $K_\infty=G'=F_2$; therefore [4], there is a number h for which $K_{L+h}=G'=F_2$, giving $L=2$, and so $S=S(a_0, a_1, a_2)$, $K_{L+h}=F(B_1, B_2)=F_2(B)$, B_1, $B_2\subset G'$.

Set $S_{L+h+1}=U$, $a_{L+h+1}=u$; form the subgroup $D=\{K_{L+h}, u; U\}$ of G. Then $D=G'$. Since u is an element of the free group $K_{L+h}=F(B)$, u has a representation as a unique B-word, $u=f(B)$; then D is given

[2] This theorem is of some interest with respect to a conjecture of Magnus that if $G=\{a_1, \cdots, a_n; R_1\}\simeq H=\{b_1, \cdots, b_n; S_1\}$, then $H=NG$.

by $\{B_1, B_2, u; \bar{u}f(B)\}$ as well as by $\{B_1, B_2, a_j, u; \bar{a}_jf_j(B), U\}$. Replace in U every $a_j \neq u$ by its representation f_j as a B-word to get $U = V(B, u)$. Then D is given by $\{B_1, B_2, u; V\}$, showing that the elements $V^{\pm 1}$ and $\bar{u}f(B)$ generate the same normal subgroup of $F_3(B_1, B_2, u)$, hence [1, p. 157] are conjugates.

U is a function of a_{h+1}, a_{h+2}, and u, with a_{h+1} and u actually occurring; for certain elements $v_i = v_i(a_{h+1}, a_{h+2}) \neq 1$, and integers x_i,

$$U = u^{x_0}v_1u^{x_1}v_2 \cdots v_ku^{x_k}.$$

In G, the elements a_{h+1}, a_{h+2}, and u are connected by the single relation U, and so a_{h+1} and a_{h+2} form a free subgroup of G [1, p. 157]. It follows that if $v_i(a_{h+1}, a_{h+2}) \neq 1$, then the representation $w_i(B)$ of v_i in terms of the free generators B_1 and B_2 of G' is not the empty word. The representation

$$V = u^{x_0}w_1 \cdots w_ku^{x_k}$$

of U as a B, u-word contains therefore the same number of u-symbols as U does. On the other hand, viewed as cyclic words, V^ϵ and $\bar{u}f(B)$ are identical, and so both contain u just once. The same must hold for $U = S_{L+h+1}$, hence for S; thus, up to conjugation and inversion, $S = \bar{a}_2f_2(a_0, a_1)$. Similarly, a_0 occurs in S just once, giving

$$S = \overset{r}{a_1}\bar{a}_2\overset{k}{a_1}\overset{\epsilon}{a_0} = (g\overset{r}{a}\bar{g})(\overset{2}{g}\bar{a}\overset{2}{\bar{g}})(g\overset{k}{a}\bar{g})\overset{\epsilon}{a}.$$

This proves the lemma.

The result above gives $a_3 = f_2(a_1, a_2) = f_3(a_0, a_1)$. The mapping of a_i onto a_{i+1} for every integer i is an automorphism A of G' given by $A(a_0) = a_1$, $A(a_1) = f_2(a_0, a_1) = a_2$, etc., so that $A^n(a_0) = a_n = f_n(a_0, a_1)$; thus the subgroup K of G' generated by a_0 and a_1 contains every a_n, is a free group $F_2(a_0, a_1)$ and contains G'. Since $K \subseteq G'$, it follows that $G' = F_2(a_0, a_1)$. This fact will be needed.

Lemma 2. *Let* $K = \{g, a, b;\ ga\bar{g} = a^{x_1}b^{y_1},\ gb\bar{g} = a^{x_2}b^{y_2},\ ab\bar{a} = b\} \simeq K^* = \{g, a, b;\ ga\bar{g} = f_1(a, b),\ gb\bar{g} = f_2(a, b,)\ ab\bar{a} = b\}$ *with*

$$K' = K^{*\prime} = F_2(a, b)/F_2'(a, b);$$

let B *be the mapping* $B(a) = a^{x_1}b^{y_1}$, $B(b) = a^{x_2}b^{y_2}$ *in* K', *with matrix* M, B^* *the mapping* $B^*(a) = f_1(a, b)$, $B^*(b) = f_2(a, b)$ *in* $K^{*\prime}$, *with matrix* M^*. *Then trace of* M *equals trace of* $M^{*\epsilon}$ *and trace of* $\overline{M}$ *equals trace of* $\overline{M}^{*\epsilon}$; $\epsilon = \pm 1$.

Proof. Because $K/K' = F_1(g) \simeq K^*/K^{*\prime}$ and the isomorphism I of K onto K^* defines an isomorphism I' of the factor commutator groups, the image $I(g)$ must be of the form $g^\epsilon h(a, b) = g^\epsilon h$, and $I(K') = K^{*\prime}$ with $I(a) = a^{r_1}b^{s_1} = \alpha$, $I(b) = a^{r_2}b^{s_2} = \beta$.

$$B^{*\epsilon}(\alpha) = g^\epsilon \alpha \bar{g}^\epsilon = g^\epsilon h \alpha \bar{h} \bar{g}^\epsilon = I(g a \bar{g}) = I(B(a)) = I(a^{x_1} b^{y_1}) = \alpha^{x_1} \beta^{y_1},$$
$$B^{*\epsilon}(\beta) = g^\epsilon \beta \bar{g}^\epsilon = g^\epsilon h \beta \bar{h} \bar{g}^\epsilon = I(g b \bar{g}) = I(B(b)) = I(a^{x_2} b^{y_2}) = \alpha^{x_2 y_2}$$

and a similar equality holds for $\overline{B}^{*\epsilon}$.

For any automorphism X and generators α, β of F_2/F_2', $X(a) = NX\overline{N}(\alpha)$, $X(b) = NX\overline{N}(\beta)$, so the equalities above show that the trace is preserved, as claimed.

Under the Nielsen transformation given by $N'(a) = a$, $N'(g) = a^k g$,

$$N'(\overset{r}{a_1} \bar{a}_2 \overset{k}{a_1} \overset{\epsilon}{a_0}) = N'(S) = \overset{\epsilon}{a} g \overset{h}{a} g \bar{a} \bar{g} \bar{g} = W, \qquad h = r + k,$$

up to an inner automorphism (Lemma 1); thus one may assume that S has the form W. Since $a \subset G' = F_2(a_0, a_1)$, the sum of its exponents in S is ± 1. This gives $h = \pm 1$ for $\epsilon = +1$, and $h = 1$ or 3 for $\epsilon = -1$. The word S is thus Nielsen image of one of the following:

$$S_1 = a g \bar{a} \bar{g} \bar{g} a g, \quad S_2 = a g \bar{a} \bar{g} \bar{g} \bar{a} g, \quad S_3 = \bar{a} \bar{g} \bar{g} \bar{a} g a^3 g, \quad S_1^* = \bar{a} g \bar{a} \bar{g} \bar{g} a g.$$

For $N(a) = a$, $N(g) = \bar{g}$, $\overline{N} S_1^* = \overline{S}_1$, so that the first three words remain to investigate. It is now established that a group G of Theorem 4 is Nielsen image of one of the groups $H_i = \{g, a; S_i\}$, i: 1, 2, 3; it remains to show that these three groups are not isomorphic. This follows from Lemma 2 if they are presented as

$$H_1^* = \{g, a, b; g a \bar{g} = b, g b \bar{g} = ab\},$$
$$H_2^* = \{g, a, b; g a \bar{g} = b, g b \bar{g} = \bar{a} b\},$$
$$H_3^* = \{g, a, b; g a \bar{g} = b, g b \bar{g} = \bar{a} b^3\}.$$

To H_i^* belongs the automorphism A_i of $F_2(a, b)$ induced by conjugation with g and the matrix M_i with trace c_i and determinant ϵ_i, where

$$(c_1, \epsilon_1) = (-1, -1); \qquad (c_2, \epsilon_2) = (+1, +1); \qquad (c_3, \epsilon_3) = (3, +1).$$

This completes the proof of Theorem 4. H_2 is the fundamental group of the (trefoil) knots with three crossings, $H_3 = L$ that of the (Listing's) knot with four crossings. H_1 does not belong to a knot, since its L-polynomial, $-x^2 - x + 1$ [7] is not symmetric in the coefficients [8].

It follows from Theorems 1 and 4 that the group L has non-Nielsen automorphisms but only Nielsen isomorphisms.

Let A_i designate automorphisms of $F_2(a, b)$.

COROLLARY 1. *If* $\{g, a, b; g a \bar{g} = A_1(a), g b \bar{g} = A_1(b)\} = G_1$, *and* $IG_1 = G_2$ *is similarly defined, and* $G_1' = F_2$, *then* $G_2 = NG_1$.

PROOF. Again the mapping I is of the form $I(g) = g^\epsilon f_1(a, b)$, $I(a)$

$=A_3(a)$, $I(b)=A_3(b)$, and these image words clearly generate g, a, and b. In particular, if G_1 is isomorphic to one of the groups H_i, i: 1, 2, or 3, of Theorem 4, then $G_1=NH_i^*$.

COROLLARY 2. *If the defining relations of*

$$G = \{g, a, b; ga\bar{g} = A(a), gb\bar{g} = A(b)\}$$

equal those of a H_i^* *of Theorem* 4 *taken modulo* $[a, b]$, *then* $NG=H_i^*$, $N(g)=\bar{w}(a, b)g$, $N(a)=a$, $N(b)=b$, *for* $w\subset F_2(a, b)$.

PROOF. If $A_1(a)=A_2(a)$ mod $[a, b]$, $A_1(b)=A_2(b)$ mod $[a, b]$, then A_1 differs from A_2 by an inner automorphism [**5**]. Thus the automorphism A of G' induced by g differs from the automorphism A^* of H_i^* induced by g by inner automorphisms of $F_2(a, b)$: $A^*(a)=wA(a)\bar{w}$, $A^*(b)=wA(b)\bar{w}$. Then, under N given above,

$$\begin{aligned} N(A(a)g\bar{a}\bar{g}) &= A(a)\bar{w}g\bar{a}\bar{g}w \\ &= \bar{w}A^*(a)g\bar{a}\bar{g}w. \end{aligned}$$

Similarly for b, so that under N the relations of G take the form of the relations of H_i^*, as claimed.

THEOREM 5. *Let* $G=\{g, a, b; ga\bar{g}=A(a), gb\bar{g}=A(b)\}$ *be extension of* $F_2(a, b)$ *by* $F_1(g)$, $A(a)=a^{x_1}b^{y_1}$ mod $[a, b]$, $A(b)=a^{x_2}b^{y_2}$ mod $[a, b]$, $x_1+y_2=c$, $x_1y_2-x_2y_1=\epsilon$. $G'=F_2(a, b)$ *if and only if* $(c, \epsilon)=(3, 1)$, $(1, 1)$, *or* $(\pm 1, -1)$.

PROOF. Let $F=F_2(a, b)/F_2'(a, b)$, $U=a^ub^y$, $Y=a^cb^d$, $ud-yc=\pm 1$, $N(a)=a^xb^z$, $N(b)=a^pb^q$, $xq-pz=\epsilon$; set $w=a^{w_a}b^{w_b}\subset F$. Then

$$\begin{aligned} (N(U)\overline{U})_a &= ux+yp-u, & (N(U)\overline{U})_b &= uz+yq-y, \\ (N(Y)\overline{Y})_a &= cx+dp-c, & (N(Y)\overline{Y})_b &= cz+dq-d. \end{aligned}$$

The elements $N(U)\overline{U}$, $N(Y)\overline{Y}$ generate F if and only if

$$\begin{aligned} \pm 1 &= (ux+yp-u)(cz+dq-d)-(uz+yq-y)(cx+dp-c) \\ &= (ud-yc)((xq-pz)+1-(x+q)) = \pm(\epsilon+1-(x+q)), \end{aligned}$$

that is if the pair of values $(x+q, \epsilon)$ is either (3, 1), or (1, 1), or $(\pm 1, -1)$. The proof is based on this fact.

If the condition on (c, ϵ) holds then $ga\bar{g}\bar{a}=A(a)\bar{a}$ and $gb\bar{g}\bar{b}=A(b)\bar{b}$ generate F, so that together with conjugates of $ab\bar{a}\bar{b}$ they generate $F_2(a, b)$. Then $F_2(a, b)\subset G'$; but since $G'\subset F_2(a, b)\subset G$, $F_2(a, b)=G'$, as claimed.

If $F_2(a, b)=G'$ then it is generated by $ga\bar{g}\bar{a}$, $gb\bar{g}\bar{b}$, $ab\bar{a}\bar{b}$ and their conjugates in G, and G'/G'' is generated by $ga\bar{g}\bar{a}=\alpha$, $gb\bar{g}\bar{b}=\beta$ and their conjugates by g^k, k: $\pm 1, \cdots$. If $ga\bar{g}=a^xb^y$, $gb\bar{g}=a^ub^v$ modulo G'', they generate G'/G'', and $\alpha=a^{x-1}b^y$, $\beta=a^ub^{v-1}$ in G'/G''. As $g\alpha\bar{g}=(a^xb^y)^{x-1}(a^ub^v)^y=(a^{x-1}b^y)^x(a^ub^{v-1})^y=\alpha^x\beta^y$, and similarly $g\beta\bar{g}=\alpha^u\beta^v$, the elements α, β already generate the factor group G'/G''. As $\alpha=A(a)\bar{a}$, $\beta=A(b)\bar{b}$, the condition on (c, ϵ) must hold, as claimed.

It may be noted that for $\epsilon=-1$, the inverse transformation $\bar{A}$ has the trace $-c$, so that only three possibilities remain modulo the Nielsen transformation $N(a)=a$, $N(b)=b$, $N(g)=\bar{g}$.

THEOREM 6. *The group G of Theorem 5 is isomorphic to one of the groups H_i^* of Theorem 4 if and only if $(c, \epsilon)=(c_i, \epsilon_i)$.*

PROOF. The necessity of the condition follows by an easy step from Lemma 2. Suppose the condition satisfied. Designate by (A) the group of all automorphisms of F_2, by (I) the group of inner automorphisms of F_2, and by (A') the group of automorphisms of F_2/F_2'; then (A') is given by the modular group and $(A')\simeq(A)/(I)$ [5]. It is known [9] that in the domain of unimodular integral two-by-two matrices (the modular group) the class number for trace c_i and determinant ϵ_i, i: 1, 2, 3, is one; hence the matrix M belonging to G (see Lemma 2) is a conjugate $NM_i\bar{N}$ of the matrix M_i belonging to H_i^*, with $|N| \pm 1$. This, together with $(A')\simeq(A)/(I)$, gives: $A=NA_i\bar{N}$ (see Lemma 2), and so $G=NH_i^*$, as claimed.

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