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FUNCTIONS HAVING POSITIVE REAL PART IN AN ELLIPSE

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1. **Introduction.** Carathéodory [1] proved that if an analytic function $f(z)$, $f(0)=1$, is regular and has positive real part in the unit circle then the nth coefficient must lie in the closed circle with center at the origin and with radius equal to 2, the extremal function being essentially $(1+z)(1-z)^{-1}$. This theorem has been used extensively in the study of various subclasses of univalent functions. It has been generalized to functions which are regular and have positive real part in an annulus by Nehari [2].

In this paper we obtain an analogous theorem for functions which are regular and have positive real part in an ellipse.

2. **The main result.** Let $f(z)$ be regular in the ellipse E with foci at ± 1 and semiaxes $a>b$, $a>1$. Such a function is representable by a series of Tchebychef polynomials which converges uniformly in E, [3; 4]. That is,

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z), \qquad z \in E \tag{2.1}$$

where $T_n(z) = \cos\,(n \arccos z)$.

E may be represented parametrically by $z=a \cos t+ib \sin t$, $0\leq t<2\pi$. Since $a^2-b^2=1$ we may write $a=\cosh s$, $b=\sinh s$, $s>0$.

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Then $z = \cosh s \cos t + i \sinh s \sin t = \cos (t - is)$ represents the ellipse E, where $s = \tanh^{-1} b/a$. Suppose $a_0 = 1$ and $a_n = \alpha_n + i\beta_n$ then

$$
\begin{aligned}
f(z) = U(s, t) + iV(s, t) &= 1 + \sum_{n=1}^{\infty} a_n T_n(\cos (t - is)) \\
&= 1 + \sum_{n=1}^{\infty} a_n \cos n(t - is).
\end{aligned} \tag{2.2}
$$

Hence

$$
\operatorname{Re} \{f(z)\} = U(s, t) = 1 + \sum_{n=1}^{\infty} \alpha_n \cosh ns \cos nt - \beta_n \sinh ns \sin nt
$$

which for fixed s is the Fourier expansion for $U(s, t)$, so that

$$
\begin{aligned}
1 &= 1/2\pi \int_0^{2\pi} U(s, t) dt, \\
\alpha_n \cosh ns &= 1/\pi \int_0^{2\pi} U(s, t) \cos nt dt, \\
\beta_n \sinh ns &= 1/\pi \int_0^{2\pi} U(s, t) \sin nt dt.
\end{aligned} \tag{2.3}
$$

It follows now that

$$
\left| \alpha_n \cosh ns + i\beta_n \sinh ns \right| = 1/\pi \left| \int_0^{2\pi} U(s, t) e^{-int} dt \right| \leqq 2. \tag{2.4}
$$

Employing the exponential forms for $\cosh ns$ and $\sinh ns$ (2.4) becomes

$$
\alpha_n^2 (R^n + R^{-n})^2 + \beta_n^2 (R^n - R^{-n})^2 \leqq 16, \tag{2.5}
$$

where $R = a + b > 1$, which means that $a_n = \alpha_n + i\beta_n$ must be inside the closed ellipse with center at the origin and semiaxes $4(R^n + R^{-n})^{-1}$ and $4(R^n - R^{-n})^{-1}$.

We now have the following

THEOREM. *Let $f(z)$ be regular and have positive real part in the ellipse E and let $f(z)$ have the expansion*

$$
f(z) = 1 + \sum_{n=1}^{\infty} a_n T_n(z), \qquad z \in E, \tag{2.6}
$$

in Tchebychef polynomials which converges uniformly in E. Further let $R = a + b > 1$ and $a_n = \alpha_n + i\beta_n$, then

$$\alpha_n^2(R^n + R^{-n})^2 + \beta_n^2(R^n - R^{-n})^2 \leqq 16. \tag{2.5'}$$

This inequality is the best possible.

It remains to show that the inequality (2.5′) is sharp. We shall do this by constructing, for each n, a function $f_n(z)$ satisfying equality in (2.5′). By the Riemann mapping theorem it is possible to map E onto the right half plane such that an arbitrary point $z_0 = \cos(t_0 - is)$ on E is mapped into the point at infinity. The mapping function will be uniquely determined up to a linear transformation of the right half plane onto itself. Let $w = f_n(z) = F(z, z_0)$, z_0 on the boundary of E, be a function mapping E onto Re $\{w\} \geqq 0$ with z_0 mapping into the point at infinity. Since $f_n(z)$ is analytic in E it has the series expansion $f_n(z) = b_0 + \sum_{n=1}^{\infty} b_k T_k(z)$.

If $b_0 = \gamma + i\delta \neq 1$ we consider the function

$$f_n^*(z) = \frac{f_n(z) - i\delta}{\gamma} = 1 + \sum_{k=1}^{\infty} a_k T_k(z)$$

where $\gamma > 0$ by (2.3). $f_n^*(z)$ has the desired normalization and Re $\{f_n^*(z)\} = 0$ almost everywhere on Γ the boundary of E, hence it suffices to assume that $f_n(z)$ has an expansion of the form (2.6).

Suppose we consider the function

$$Q(z) = 1 - \left[T_k(z_0)T_k(z) + U_{k-1}(z_0)U_{k-1}(z)(1 - z_0^2)^{1/2}(1 - z^2)^{1/2}\right]$$

where $U_{k-1}(z) = \sin k(t - is)/(1 - z^2)^{1/2}$ is a Tchebychef polynomial of the second kind. It is easy to show that $Q(z)$ is non-negative on Γ and also that $Q(z_0) = 0$. Since $f_n(z)$ has a simple pole at $z = z_0$ it follows that $f_n(z)\ Q(z)$ is regular on Γ.

Next consider the integral

$$I = -1/\pi \int_\Gamma f_n(z)Q(z) \frac{dz}{(1 - z^2)^{1/2}} \cdot \tag{2.7}$$

Using the orthogonal properties of the Tchebychef polynomials (i.e., $\int_\Gamma T_m(z)T_n(z)(1 - z^2)^{-1/2}dz = 0,\ -\pi,\ -2\pi$ if $m \neq n$, $m = n \neq 0$, $m = n = 0$, respectively) and $\int_\Gamma T_m(z)U_{n-1}(z)dz = 0$ for all integral $m \geqq 0$, $n \geqq 1$ we have

$$\text{Re}\ \{I\} = \text{Re}\ \{2 - T_n(z_0)a_n\}. \tag{2.8}$$

Now Re $\{f_n(z)\} = 0$ on Γ for $t \neq t_0$ and since the integrand in (2.7) is a continuous function of t for $0 \leqq t \leqq 2\pi$ it follows that the integral vanishes identically. Thus

$$\text{Re}\ \{T_n(z_0)a_n\} = 2 \tag{2.9}$$

or $\operatorname{Re}\{(\alpha_n \cosh ns + i\beta_n \sinh ns)e^{int_0}\} = 2$ and for a proper choice of t_0 we have

$$|\alpha_n \cosh ns + i\beta_n \sinh ns| = 2 \tag{2.10}$$

which establishes (2.5′).

We have assumed that $f(z)$ is regular on Γ in order to carry out the integration. This assumption may be relinquished by considering the function $f_\epsilon(z) = f(\cos(t - i(s-\epsilon)))$ in the ellipse for which $a = \cosh(s-\epsilon)$, $b = \sinh(s-\epsilon)$. If (2.5) is proved for $f_\epsilon(z)$ then the corresponding result for $f(z)$ follows by letting $\epsilon \to 0$.

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