YET ANOTHER NOTE ON PARACOMPACT SPACES

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1. **Introduction.** According to the usual definition [1, p. 66], a Hausdorff space X is *paracompact* if

(P_0) *Every open covering of X has an open locally finite refinement.* Several alternate definitions have been obtained during the past decade. Thus A. H. Stone showed [6, Theorem 1] that paracompactness is equivalent to *full normality*; that is, X is a T_1-space, and

(P_1) *Every open covering of X has an open star-refinement.*[2] In a different direction, the author showed [4, Theorem 1] that, in a regular space, (P_0) is equivalent to the following apparently weaker.

(P_2) *Every open covering of X has a closure-preserving*[3] *refinement.* Our principal purpose in this paper is to obtain yet another characterization of paracompactness which, in any regular space, is an easy consequence of both (P_1) and (P_2).

If $\mathcal{U}$ and $\mathcal{V}$ are collections of subsets of X, then we say thar $\mathcal{V}$ is *cushioned in* $\mathcal{U}$ if one can assign to each $V \in \mathcal{V}$ a $U_V \in \mathcal{U}$ such that, for every $\mathcal{V}' \subset \mathcal{V}$,

$$(\bigcup\{V \mid V \in \mathcal{V}'\})^- \subset \bigcup\{U_V \mid V \in \mathcal{V}'\}.$$

A refinement of $\mathcal{U}$ which is cushioned in $\mathcal{U}$ is called a *cushioned refinement* of $\mathcal{U}$. As examples of cushioned refinements of an open covering $\mathcal{U}$, let us mention an open star-refinement of $\mathcal{U}$, as well as a closure-preserving refinement of any open covering $\mathcal{W}$ which has the property that $\{\overline{W} \mid W \in \mathcal{W}\}$ refines $\mathcal{U}$ (such a $\mathcal{W}$ must exist if X is regular). We now have

THEOREM 1.1. *A T_1-space X is paracompact if and only if*

(P_3) *Every open covering of X has a cushioned refinement.*

In §2 we obtain some properties of open coverings which are easily

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[2] $\mathcal{V}$ is a *star-refinement* of $\mathcal{U}$ if, for every $V_0 \in \mathcal{V}$, $\bigcup\{V \in \mathcal{V} \mid V \cap V_0 \neq \emptyset\}$ is a subset of some $U \in \mathcal{U}$. While (P_1) does not follow *obviously* from (P_0) in a Hausdorff space, it is nevertheless a fairly easy consequence of it; the difficult implication runs in the opposite direction.

[3] $\mathcal{V}$ is *closure-preserving* if, for every $\mathcal{V}' \subset \mathcal{V}$, $(\bigcup\{V \mid V \in \mathcal{V}'\})^- = \bigcup\{\overline{V} \mid V \in \mathcal{V}\}$. Any locally finite $\mathcal{V}$ is clearly closure-preserving.

equivalent to the property of having a cushioned refinement; one of these equivalences sheds some further light on the extent to which (P_3) is apparently weaker than full normality. The proof of Theorem 1.1, which closely parallels that of [**4**, Theorem 1], is found in §3. In §4 we use Theorem 1.1 to prove the following result, which generalizes [**4**, Theorem 2]; we call $\mathcal{V}$ a *σ-cushioned refinement* of $\mathcal{U}$ if $\mathcal{V}=\bigcup_{i=1}^{\infty}\mathcal{V}_i$, with each $\mathcal{V}_i$ cushioned in $\mathcal{U}$.

THEOREM 1.2. *A T_1-space X is paracompact if and only if*
(P_3') *Every open covering of X has an open σ-cushioned refinement.*

In §5, finally, we show how Theorem 1.2 provides a simplified proof of a beautiful metrization theorem recently obtained by J. Nagata [**5**, Theorem 1].

2. **Some equivalent properties.** In the following proposition, a set $V\subset X\times X$ is called a *semi-neighborhood of the diagonal* if both $V(x)$ and[4] $V^{-1}(x)$ are neighborhoods of x for every $x\in X$.

PROPOSITION 2.1. *The following properties of an open covering $\mathcal{U}$ of a topological space X are equivalent.*

(a) *$\mathcal{U}$ has a cushioned refinement $\mathcal{V}$.*

(b) *There exists an indexed covering $\{W_U \mid U\in\mathcal{U}\}$ of X such that $(\bigcup\{W_U\mid U\in\mathcal{U}'\})^-\subset\bigcup\{U\mid U\in\mathcal{U}'\}$ for every $\mathcal{U}'\subset\mathcal{U}$.*

(c) *One can assign to each $x\in X$ a $U_x\in\mathcal{U}$ such that, for every $X'\subset X$,* we have $\overline{X'}\subset\bigcup\{U_x\mid x\in X'\}$.

(d) *There exists a semi-neighborhood V of the diagonal such that $\{V(x)\mid x\in X\}$ refines $\mathcal{U}$.*

PROOF. It suffices to prove the following implications:

(a)→(b). Let $W_U=\bigcup\{V\in\mathcal{V}\mid U_V=U\}$.

(b)→(c). Pick U_x such that $x\in W_{U_x}$.

(c)→(a). Let $\mathcal{V}=\{\{x\}\mid x\in X\}$, and let $U_{\{x\}}=U_x$.

(c)→(d). Let $V=\{(x, y)\in X\times X\mid y\in U_x\}$. Then $V(x)=U_x$, which is a neighborhood of x. Moreover, if we let $R_x=\{y\in X\mid x\notin U_y\}$, then by assumption $x\in(X-\overline{R}_x)\subset(X-R_x)=\{y\in X\mid x\in U_y\}=V^{-1}(x)$, and hence $V^{-1}(x)$ is also a neighborhood of x.

(d)→(c). Pick $U_x\in\mathcal{U}$ such that $V(x)\subset U_x$. If $X'\subset X$ and $y\in\overline{X'}$, then $V^{-1}(y)$ intersects X', and hence, for some $x\in X'$, $x\in V^{-1}(y)$ and thus $y\in V(x)$. This completes the proof.

In conclusion, let us recall that J. L. Kelley [**2**, p. 155] calls an open covering *even* if it satisfies 2.1(d) with "semi-neighborhood" replaced by "neighborhood." Since Kelley showed [**2**, p. 170, U] that

[4] As usual, $V(x)=\{y\in X\mid (x, y)\in V\}$ and $V^{-1}(x)=\{y\in X\mid (y, x)\in V\}$.

the requirement that every open covering be even is easily equivalent to full normality, it follows from Proposition 2.1 that, roughly speaking, (P_3) is related to full normality as semi-neighborhoods of the diagonal are related to neighborhoods.

3. **Proof of Theorem** 1.1. To prove the nontrivial part of the theorem, we assume that the T_1-space X satisfies condition (P_3) in the statement of Theorem 1.1, and will prove that X is paracompact. Since we shall be dealing with indexed coverings, let us make the convention that an *indexed* covering $\{C_\alpha\}_{\alpha\in A}$ is a *cushioned refinement* of an *indexed* covering $\{U_\alpha\}_{\alpha\in A}$ if, for every $A'\subset A$,

$$\left(\bigcup_{\alpha\in A'} C_\alpha\right)^- \subset \bigcup_{\alpha\in A'} U_\alpha.$$

As an immediate consequence of Proposition 2.1, (a)→(b), we now have

LEMMA 3.1. *Every indexed open covering $\{U_\alpha\}_{\alpha\in A}$ of X has an indexed cushioned refinement $\{C_\alpha\}_{\alpha\in A}$.*

Using Lemma 3.1, we next prove

LEMMA 3.2. *X is normal.*

PROOF. Let E_1, E_2 be disjoint, closed subsets of X. Then $\{X-E_1, X-E_2\}$ is an open covering of X, so by Lemma 3.1 there exists a covering $\{C_1, C_2\}$ of X such that $\overline{C}_i\subset X-E_i$ for $i=1, 2$. But then the open sets $X-\overline{C}_1$ and $X-\overline{C}_2$ separate E_1 and E_2, and the proof is complete.

Since Lemma 3.2 implies that X is regular, we can now prove that X is paracompact by showing that every open covering of X has an open σ-discrete[5] refinement [4, Proposition 1].

After these preliminaries, let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of X, which has been indexed by a *well-ordered* index set A. We must show that this covering has a σ-discrete open refinement.[5]

LEMMA 3.3. *For each positive integer i, there exists a cushioned refinement $\{C_{\alpha,i}\}_{\alpha\in A}$ of $\{U_\alpha\}_{\alpha\in A}$ such that, for all α and i,*

(a) $(\bigcup_{\beta<\alpha} C_{\beta,i})^-\cap C_{\alpha,i+1}=\emptyset$,

(b) $C_{\alpha,i}\cap(\bigcup_{\beta>\alpha} C_{\beta,i+1})^-=\emptyset$.

PROOF. Let $\{C_{\alpha,1}\}_{\alpha\in A}$ be any cushioned refinement of $\{U_\alpha\}_{\alpha\in A}$

[5] $\mathcal{W}$ is *discrete* if every $x\in X$ has a neighborhood intersecting at most one $W\in\mathcal{W}$; $\mathcal{W}$ is *σ-discrete* if $\mathcal{W}=\bigcup_{i=1}^{\infty}\mathcal{W}_i$, with each $\mathcal{W}_i$ discrete.

(Lemma 3.1). Suppose that suitable refinements $\{C_{\alpha,i}\}_{\alpha\in A}$ have been found for $i=1, \cdots, n$, and let us construct $\{C_{\alpha,n+1}\}_{\alpha\in A}$. For all α, let

(1) $$U_{\alpha,n+1} = U_\alpha - \left(\bigcup_{\beta<\alpha} C_{\beta,n}\right)^-.$$

Then $\{U_{\alpha,n+1}\}_{\alpha\in A}$ is an (open) covering of X, because $x\in X$ implies $x\in U_{\alpha,n+1}$ for the *first* α for which $x\in U_\alpha$ (since, by assumption, $(\bigcup_{\beta<\alpha} C_{\beta,n})^-\subset\bigcup_{\beta<\alpha} U_\beta$). We now use Lemma 3.1 to pick a cushioned refinement $\{C_{\alpha,n+1}\}_{\alpha\in A}$ of $\{U_{\alpha,n+1}\}_{\alpha\in A}$. Then (a) follows at once from (1) and the fact that $C_{\alpha,n+1}\subset U_{\alpha,n+1}$. To see (b), note that, by (1), $C_{\alpha,n}$ is disjoint from $U_{\beta,n+1}$ for all $\beta>\alpha$, and hence from $(\bigcup_{\beta>\alpha} C_{\beta,n+1})^-$ $\subset\bigcup_{\beta>\alpha} U_{\beta,n+1}$.

LEMMA 3.4. *There exists an indexed open covering*

$$\{V_{\alpha,i} \mid \alpha \in A, i = 1, 2, \cdots\}$$

of X *such that, for all* i,

(a) $V_{\alpha,i}\subset U_\alpha$ *for all* α,

(b) $V_{\alpha,i}\cap V_{\beta,i}=\emptyset$ *whenever* $\alpha\neq\beta$.

PROOF. For each α and i, let

$$V_{\alpha,i} = X - \left(\bigcup_{\beta\neq\alpha} C_{\beta,i}\right)^-.$$

Since $\{C_{\alpha,i}\}_{\alpha\in A}$ is a covering of X for each i, we have

$$V_{\alpha,i} \subset C_{\alpha,i} \subset U_\alpha$$

for all α and i, which proves (a) and (b). Since each $V_{\alpha,i}$ is clearly open, it remains to show that the sets $V_{\alpha,i}$ cover X.

Pick an $x\in X$, and let us find a $V_{\alpha,i}$ containing it. Using the well-ordering of the index set A, let

$$\alpha_i = \min\{\alpha \in A \mid x \in C_{\alpha,i}\} \qquad i = 1, 2, \cdots,$$

and then pick a positive integer k such that

$$\alpha_k = \min\{\alpha_i \mid i = 1, 2, \cdots\}.$$

Let us show that

$$x \in V_{\alpha_k,k+1}.$$

Note first that $x\in C_{\alpha_k,k}$ by definition of α_k, and hence, by 3.3(b) (with $i=k$),

(2) $$x \notin \left(\bigcup_{\alpha>\alpha_k} C_{\alpha,k+1}\right)^-.$$

Again from the definition of α_k, we have $x \in C_{\alpha,k+2}$ for some $\alpha \geqq \alpha_k$, and hence

$$x \notin \left(\bigcup_{\beta < \alpha_k} C_{\beta,k+1} \right)^- \tag{3}$$

by 3.3(a) with $i = k+1$. It follows from (2) and (3) that $x \in V_{\alpha_k,k+1}$, which completes the proof of the lemma.

To complete the proof of Theorem 1.1, we apply Lemma 3.1 once more to obtain a cushioned refinement $\{D_{\alpha,i} \mid \alpha \in A, i = 1, 2, \cdots\}$ of $\{V_{\alpha,i} \mid \alpha \in A, i = 1, 2, \cdots\}$. Now for each i, $(\bigcup_{\alpha\in A} D_{\alpha,i})^- \subset \bigcup_{\alpha \in A} V_{\alpha,i}$ and hence, remembering that X is normal (Lemma 3.1), there exists an open $G_i \subset X$ such that $(\bigcup_{\alpha\in A} D_{\alpha,i})^- \subset G_i \subset \bar{G}_i \subset \bigcup_{\alpha\in A} V_{\alpha,i}$. Letting

$$\mathcal{W}_i = \{V_{\alpha,i} \cap G_i \mid \alpha \in A\} \qquad i = 1, 2, \cdots,$$

we see that each $\mathcal{W}_i$ is a discrete family of open sets, and that $\bigcup_{i=1}^{\infty} \mathcal{W}_i$ is the required σ-discrete open refinement of $\{U_\alpha\}_{\alpha\in A}$.

4. **Proof of Theorem** 1.2. Let us begin by noting that 2.1(c) can be rephrased as follows:

2.1(c′). One can assign to each $x \in X$ a $U_x \in \mathcal{U}$ containing x, and to each $y \in X$ a neighborhood W_y of y, such that $y \notin U_x$ implies $x \notin W_y$.

Suppose now that the open covering $\mathcal{U}$ of X has a σ-cushioned refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ (with the $U \in \mathcal{U}$ assigned to a $V \in \mathcal{V}_n$ denoted by $U_{V,n}$), and let us show that $\mathcal{U}$ satisfies 2.1(c′). For each $x \in X$, let $n(x) = \inf \{n \mid x \in \bigcup \mathcal{V}_n\}$, pick a $V(x) \in \mathcal{V}_{n(x)}$ which contains x, and let $U_x = U_{V(x),n(x)}$. For $y \in Y$, pick any index $m(y)$ such that $y \in \bigcup \mathcal{V}_{m(y)}$, and let

$$W_y = \bigcup \mathcal{V}_{m(y)} - \bigcup_{k=1}^{m(y)} (\bigcup \{V \in \mathcal{V}_k \mid y \notin U_{V,k}\});$$

then W_y is a neighborhood of y, because each $\mathcal{V}_k$ is cushioned in $\mathcal{U}$. To see that 2.1(c′) is satisfied, suppose that $y \notin U_x$. If $n(x) \leqq m(y)$, then $V(x)$ has been subtracted out from W_y, and hence $x \notin W_y$. If $n(x) > m(y)$, then $x \notin \bigcup \mathcal{V}_{m(y)}$ by definition of $n(x)$, and again $x \notin W_y$. This completes the proof.

The above proof shows that, in Theorem 1.2, the condition that the σ-cushioned refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ be open can be weakened to requiring only that the interiors of the sets $\bigcup \mathcal{V}_n$ $(n = 1, 2, \cdots)$ cover X; one thus obtains a result which simultaneously generalizes Theorems 1.1 and 1.2. Going a step further, it is even sufficient to require only that, for each $y \in Y$, the set

$$W_y = \bigcup_{m=1}^{\infty} \left(\bigcup \mathcal{V}_m - \bigcup_{k=1}^{m} (\bigcup \{V \in \mathcal{V}_k \mid y \notin U_{V,k}\}) \right)$$

is a neighborhood of y. This last and (so far) weakest characterization of paracompactness will be applied elsewhere.

5. **An application of Theorem** 1.2. The following metrization theorem was recently proved by J. Nagata [5].

THEOREM 5.1 (J. NAGATA). *For a T_1-space X to be metrizable, it is necessary and sufficient that every $x \in X$ have (not necessarily open) neighborhoods $S_n(x)$ and $U_n(x)$ $(n=1, 2, \cdots)$, with $\{U_n(x)\}_{n=1}^{\infty}$ a local base at x, such that*

(a) *$y \notin U_n(x)$ implies $S_n(y) \cap S_n(x) = \emptyset$.*

(b) *$y \in S_n(x)$ implies $S_n(y) \subset U_n(x)$.*

Necessity is obvious; the main step in Nagata's proof of sufficiency is to show (without using (b)) that X must be paracompact, after which metrizability follows easily from previously known metrization theorems. Nagata's proof of paracompactness employs an ingenious and intricate transfinite construction; our purpose in this section is to use Theorem 1.2 to give a very simple proof (also without assuming (b)).

Let $\mathcal{W}$ be an open covering of X. For each n, let[6]

$$\mathcal{V}_n = \{S_n^0(x) \mid U_n(x) \subset W \text{ for some } W \in \mathcal{W}\},$$

and let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$. Then $\mathcal{V}$ is an open covering of X, and it only remains to check that each $\mathcal{V}_n$ is cushioned in $\mathcal{W}$. For each $V \in \mathcal{V}_n$, pick a $W_V \in \mathcal{W}$ such that, for some $x \in X$, $V = S_n^0(x)$ and $U_n(x) \subset W_V$. To see that this works, let $\mathcal{V}' \subset \mathcal{V}_n$, and let $y \notin \bigcup \{W_V \mid V \in \mathcal{V}'\}$; it then follows from (a) that $S_n(y) \cap V = \emptyset$ for all $V \in \mathcal{V}'$, and hence $y \notin (\bigcup\{V \mid V \in \mathcal{V}'\})^-$. This completes the proof.

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[6] $S_n^0(x)$ will denote the interior of $S_n(x)$.