MAPPING AND SPACE RELATIONS[1]

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1. **Introduction.** Let $f(X)=Y$ where X, Y are topological spaces and f is a continuous mapping. The following type theorem will be considered: Given assigned properties to X and f, what further property of f is equivalent to a specified property of Y. Known examples of theorems of this type are due to G. T. Whyburn [**6**, Theorem 2.3], A. H. Stone [**5**, Theorem 1], and the author [**4**]. This paper will establish some further examples, including some with more elementary space properties than have been previously considered.

2. **Definitions.** The space X is an *M space* provided each sequence in X converges to at most one point, and X is an *E space* provided every set in X has sequences converging to each of its limit points. The mapping f is *compact* provided $f^{-1}(C)$ is compact for each compact set C in Y. For definitions of P_1, P_2, and *semi-closed mappings* see [**4**].

3. M, E **spaces.**

THEOREM 1. *If X is an M, E space and f is quasi-compact, then Y is an M space if and only if f is semi-closed.*

PROOF. Assume f is semi-closed. Then Y is a T_1 space. If Y is not an M space then there is a sequence (y_i) in Y converging to both y and y' where $y\neq y'$, and the y_i are all distinct elements of $Y-(y+y')$. It follows that $A=f^{-1}(\sum y_i)$ is not closed since f is quasi-compact. Let $x\in\overline{A}-A$ and choose (x_j) in A where $(x_j)\rightarrow x$. If $f(x_j)=y_i$ for some i and infinitely many j, then $f(x)=y_i$ and $x\in A$ which is impossible. Therefore $B=\sum f(x_j)+f(x)$ is infinite, and is also closed since f is semi-closed. Since either $y\in\overline{B}-B$ or $y'\in\overline{B}-B$, there is a contradiction.

Next assume Y is an M space and suppose f is not semi-closed. Let C be a compact set in X where $f(C)$ is not closed in Y. Then $C_f=f^{-1}f(C)$ is not closed in X since f is quasi-compact. Let $x\in\overline{C}_f-C_f$

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and choose (x_i) in C_f where $(x_i)\to x$ and the $f(x_i)$ are distinct. Let $x_i' \in C\cdot f^{-1}f(x_i)$. Then $\sum x_i'$ has a limit point x' in C. Since X is an E space, some subsequence (x_{i_j}') of (x_i') converges to x'. But $f(x_{i_j}')$ $=f(x_{i_j})$ and hence $(f(x_{i_j}))\to f(x')$. Since $f(x)\notin f(C)$ and $f(x')\in f(C)$, this contradicts the assumption that Y is an M space.

THEOREM 2. *If X is an M, E space and f is quasi-compact, then Y is an M, E space if and only if f is a semi-closed and P_1 mapping.*

PROOF. Assume f is a semi-closed and P_1 mapping. Let $y\in\overline{A}$ $-A\subset Y$ and choose $x\in f^{-1}(y)\cdot \mathrm{Cl}\, f^{-1}(A)$ [**4**, Lemma 1]. Let $(x_i)\to x$ where the $x_i\in f^{-1}(A)$. Then $f(x_i)\to y$ and $f(x_i)\in A$. This proves Y is an E space. Now Y is an M space by Theorem 1.

Next assume Y is an M, E space and suppose f is not a P_1 mapping. Then there is a $y\in Y$ and a neighborhood U of $f^{-1}(y)$ such that $y\notin \mathrm{int}\, f(U)$. Choose a sequence (y_i) in $Y-f(U)$ such that $(y_i)\to y$. Now $A=\sum y_i+y$ is closed. Hence $f^{-1}(A)\cdot(X-U)=f^{-1}(\sum y_i)$ is a closed inverse set. Therefore $\sum y_i$ is closed and this is impossible. Hence f is a P_1 mapping. Now f is semi-closed by Theorem 1.

4. **Metric spaces.**

LEMMA 1. *If Y is a T_1 space, f is a P_1 mapping and* $\mathrm{Fr}\, f^{-1}(y)$ *is compact for each $y\in Y$, then f is a P_2 mapping.*

PROOF. First let $y\in Y$ where $C=\mathrm{Fr}\, f^{-1}(y)\neq 0$. It will be shown that if U is a neighborhood of C, then $y\in \mathrm{int}\, f(U)$. Let $V=U+\mathrm{int}\, f^{-1}(y)$. Then V is a neighborhood of $f^{-1}(y)$ and $f(U)=f(V)$. Hence $y\in\mathrm{int}\, f(V)$ $=\mathrm{int}\, f(U)$ since f is a P_1 mapping. Next assume $y\in Y$ where $\mathrm{Fr}\, f^{-1}(y)$ $=0$. Then $f^{-1}(y)$ is an open inverse set in X and hence $\{y\}$ is open in Y. If $x\in f^{-1}(y)$ then $\{x\}$ is compact and $y\in\mathrm{int}\, f(U)$ for each neighborhood U of x. Therefore f is a P_2 mapping.

THEOREM 3. *If X is a metric space and f is closed, then Y is a metric space if and only if f is a P_2 mapping.*

PROOF. Assume Y is a metric space. Then $\mathrm{Fr}\, f^{-1}(y)$ is compact for each $y\in Y$ [**5**, Theorem 1] and hence f is a P_2 mapping by Lemma 1.

Assume f is a P_2 mapping. Then Y has a locally countable basis [**4**, Lemma 3] and is hence a metric space [**5**, Theorem 1].

Since open mappings are clearly P_2 mappings, Theorem 3 is an evident generalization of a theorem of Balachandran [**1**].

LEMMA 2. *If X, Y are M, E spaces and f is compact, then f is closed.*

PROOF. Suppose f is not closed and let C be closed in X, $y\in\mathrm{Cl}\, f(C)$ $-f(C)$. There is a sequence (y_i) in $f(C)$ such that $(y_i)\to y$. Then

$f^{-1}(\sum y_i + y)$ is compact, and if $x_i \in C \cdot f^{-1}(y_i)$ then $\sum x_i$ has a limit point $x \in C$. Also $f(x) = y$ by the continuity of f since Y is an M space. Hence $y \in f(C)$ and this gives a contradiction. Therefore f is closed.

THEOREM 4. *If X is a metric space and f is compact, then Y is a metric space if and only if f is a P_2 mapping.*

PROOF. Assume f is a P_2 mapping. Then Y is a T_1 space since f is quasi-compact and if $y \in Y$ then $y = ff^{-1}(y)$ where $f^{-1}(y)$ is compact and hence closed. Suppose Y is not an M space and choose (y_i) in Y converging to both y and y' where y, y', y_i are distinct. If $x_i \in f^{-1}(y_i)$ then $\sum x_i$ has a limit point x since f is compact, and $y = f(x) = y'$ by the continuity of f. This gives a contradiction and Y is therefore an M space. Thus Y is an M, E space by Theorems 1 and 2. Hence Y is a metric space by Lemma 2 and Theorem 3.

If Y is assumed to be a metric space, then f is closed by Lemma 2 and hence a P_2 mapping by Theorem 3.

5. **Decomposition spaces.** A decomposition G of X is *canonical* [3] if whenever A, A_i are elements of the decomposition and $A \cdot \lim\inf A_i \neq 0$ then $A \supset \lim\sup A_i$. It follows from the proof of a theorem due to G. T. Whyburn [6, Theorem 1.1] that for X an M, E space this property is equivalent to C_g being closed whenever C is compact (the subscript "g" denoting the union of elements of G which intersect C). It is clear that the latter property is equivalent to the natural mapping g of X onto the resulting decomposition space X' being semi-closed. It therefore follows from Theorem 1:

THEOREM 5. *Let G be a decomposition of an M, E space X and let X', g denote the resulting decomposition space and natural mapping. The following are equivalent*:

(1) *The space X' is an M space.*
(2) *The mapping g is semi-closed.*
(3) *The decomposition G is canonical.*

A decomposition G of X is *quasi-continuous* [2] provided whenever A is an element of G and U is a neighborhood of A_g then there is an open set $V \subset X$ such that $A \subset V = V_g \subset U_g$. It is easily established that G is quasi-continuous if and only if g is a P_1 mapping. It therefore follows from Theorem 2:

THEOREM 6. *With the hypotheses of Theorem 5 the following are equivalent*:

(1) *The space X' is an M, E space.*

(2) *The mapping g is semi-closed and* P_1.

(3) *The decomposition G is canonical and quasi-continuous.*

6. **An example.** Let X be the polar plane and let $A_1=(1, \pi/2)$ and A_i denote the closed line segment with end points $(1, \pi/2i)$ and $(1/i, \pi/2i)$, $i=2, 3, \cdots$. Let the elements of G be A_i, $i=1, 2, \cdots$ and the points in $X-\sum A_i$. If p denotes the pole and $U=\{(\rho, \theta): \rho<1/2\}$, then $U_g=U+\sum_{i=3}^{\infty} A_i$. If V is a neighborhood of p in X such that $V=V_g$, then V contains a neighborhood of A_i for some i and is hence not contained in U_g. Therefore G is not quasi-continuous, nor is it canonical by consideration of the sequence $A_1, A_2, \cdots$. However if $\{(\rho, 0): \rho>0\}$ is deleted from X and other elements of G remain the same, then the decomposition is canonical but still not quasi-continuous.

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