ON A THEOREM OF HERSTEIN

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Let A be a simple ring which is not a 4-dimensional algebra over a field of characteristic 2, and let U be a Lie ideal of A. Then Herstein proves in [2] that either U is contained in the center Z of A, or U contains $[A, A]$; if in addition U is also assumed to be an associative ring and contains $[A, A]$, then $U=A$.

Herstein and Baxter also prove in [1; 3], and [4] that $[A, A]$ mod $[A, A]\cap Z$ is a simple Lie ring. Our object here is to give a simple proof of this fact, basing our argument only on the results of [2] quoted above, though many of Herstein's ideas will be incorporated here without explicit mention.

We first recall a few definitions and make some preliminary remarks. If u and a are elements of the ring A, $[u, a]$ will designate $ua-au$, and $[U, A]$ will denote the module generated by all $[u, a]$ where u is in U and a in A. U is an ideal of A if U is a module and if $[U, A]$ is contained in U. If U and V are submodules of A then $U+V$ is the module generated by U and V. If U is a submodule of A we shall define $U{:}A$ to be the set of elements x of A such that $[x, A]\subset U$, thus $[U{:}A, A]$ is contained in U. It is easy to check that $U{:}A$ is a ring from the identity $[uv, a]=[u, va]+[v, au]$. When U is a Lie ideal of A then $U{:}A$ is also such.

The following identities will be needed in the sequel; they are easy to verify:

$$(*) \qquad [u, v]a = [u, va] - v[u, a],$$

$$(**) \qquad b[u, v]a = [u, v]ab + [b, [u, v]a].$$

We note the following facts about U when U is a Lie ideal of $[A, A]$; we shall let S stand for $U{:}A$ whence $[S, A]\subset U$ and let $T=[S, S]$.

(1) If $U=U_0$, and for n a natural number, U_n is defined to be $U_{n-1}+[U_{n-1}, A]$ then U_n is a Lie ideal of $[A, A]$.

(2) If $V=\sum_{n=1}^{\infty} U_n$ then V is a Lie ideal of A.

(3) $U_{n+1}{:}A\supset U_n$ and $U_{n+1}{:}A\supset U_n{:}A$.

(4) $[U, U]\subset S$; for $[[U, U]A]$ is contained in $[[U, A]U]$ by the Jacobi identity.

(5) $[U, S]\subset S$; for $[[U, S]A]\subset[[U, A]S]+[[S, A]U]$.

(6) $[[A, S]S]\subset S$. This follows from (5) and the definition of S.

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(7) $[[S, S]A]\subset S$. This follows from (6) and the Jacobi identity.

(8) $[T, T]A\subset S$ and $A[T, T]\subset S$. The first statement follows from (*) and (7) with u and v in $T=[S, S]$ and a arbitrary in A. The second statement requires a similar argument.

(9) $A[T, T]A\subset S+U$. This follows from (**) and (8) with u, v in T, and a, b arbitrary in A.

(10) $[S+U, S+U]\subset S$ because of (4) and (5).

These preliminaries are sufficient to prove the theorem.

THEOREM. *If A is a non-Abelian simple ring which is not a 4 dimensional algebra over a field of characteristic 2 and if U is a proper Lie ideal of $[A, A]$ then U is contained in Z, the center of A.*

PROOF. If $S=U:A$ then S is properly contained in A; for $[S, A]\subset U$ and by hypothesis U is a proper Lie ideal of $[A, A]$. If T designates $[S, S]$ again, then the ideal of A generated by $[T, T]$ is contained in $S+U$ by (8) and (9). If $S+U$ were equal to A, then by (10) S would be a Lie ideal of A as well as a subring; and hence by the result of [**2**] quoted at the outset S would be in Z, whence U would be a Lie ideal of $A=S+U$, and hence U would be in Z as the theorem asserts.

If now U were assumed not to be in Z, then $S+U$ is properly contained in A, whence $[T, T]=0$ since A is simple. This means that $[S, S]$ is Abelian. Since U is not in Z, U_n as defined in (1) is not in Z for all n and if S_n denotes $U_n:A$ then by the above argument $[S_n, S_n]$ is Abelian. But V as defined in (2) is a Lie ideal of A not contained in Z and hence $V=[A, A]$. We note from (3) that $S_n\subset S_{n+1}$ and let Q denote $\sum_{n=1}^{\infty} S_n$. Then Q is a ring which is also a Lie ideal since Q contains $[A, A]$ by (3) and (2). Hence $Q=A$ by [**2**]. But $[Q, Q]$ is Abelian since $[S_n, S_n]$ is, and hence $[A, A]$ is Abelian.

Since $[A, A]$ is Abelian the centralizer of A is a ring which contains $[A, A]$ and consequently is a Lie ideal of $[A, A]$. Hence the centralizer of $[A, A]$ is all of A and $[A, A]\subset Z$. Now if x and y are in A, then $0=[[x, yx]y]=[[x, y]x, y]=[x, y]^2$ where $[x, y]$ is in Z. But then $[x, y]A$ is an ideal of A not equal to A since A is non-Abelian and $[x, y]A$ is a zero ring. It follows that $[x, y]A=0$ and $([x, y])$ is an ideal of A. Again we must conclude $[x, y]=0$ and hence A is Abelian contrary to hypothesis. Our assumption that U is not in Z leads to a contradiction and the theorem is proved.

REMARK. If U is a subring and also a Lie ideal of a ring A then the ideal generated by $[U, U]$ is contained in U. For if u and v are in U and a, b in A then it follows from (*) that $[U, U]A$ is contained in U and similarly $A[U, U]$ is in U; furthermore from (**) it follows that $A[U, U]A$ is in U.

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RATIONAL APPROXIMATION TO SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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Introduction. It was observed by Liouville (C. R. Acad. Sci. Paris, vol. 18 (1844) pp. 910–911; J. Math. Pures Appl. vol. 16 (1851) pp. 133–142) that an element α of the field $\boldsymbol{C}$ of complex numbers which is algebraic of degree n (over the ring $\boldsymbol{Z}$ of rational integers) can not be approximated very well by rational numbers, in the following sense: there exists a real number $\gamma>0$ such that $|\alpha-p/q|\geqq\gamma/|q|^n$ for all p, $q\in\boldsymbol{Z}$ with $q\neq 0$ and $p/q\neq\alpha$. Using this theorem Liouville gave the first examples of transcendental numbers. The proof depends only on the circumstance that every nonzero element of $\boldsymbol{Z}$ has absolute value $\geqq 1$ and the following two obvious facts (in the statement of which f denotes the polynomial of degree n vanishing at α): (i) α is an isolated point of the set of zeros of f; (ii) $f(y/z)$ can be written as a fraction in which the numerator is a polynomial in y and z and the denominator is z^n. It follows that Liouville's theorem has an abstract version in which $\boldsymbol{C}$ and $\boldsymbol{Z}$ are replaced by an arbitrary nontrivially valued field and a nonzero subring thereof in which each nonzero element has value $\geqq 1$. Since the field $K((X^{-1}))$ of power series in the reciprocal of an indeterminate X over a given commutative field K admits a valuation for which the series $u=c_mX^{-m}+c_{m+1}X^{-(m+1)}+\cdots$ (with $c_m\neq 0$) has the value $|u|=e^{-m}$, and the polynomial ring $K[X]$ is a subring of $K((X^{-1}))$ in which every nonzero element has value $\geqq 1$, Liouville's theorem applies in this situa-

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