A COMPARISON THEOREM FOR ELLIPTIC EQUATIONS[1]

M. H. PROTTER

Hartman and Wintner [1] obtained a Sturmian comparison theorem for self-adjoint second order linear elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + fu = 0, \qquad a_{ij} = a_{ji} \tag{1}$$

in a bounded domain D with boundary Γ. It is the purpose of this note to extend their result to general second order linear elliptic equations. Let v be a solution of another elliptic equation of the same form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\alpha_{ij}\frac{\partial v}{\partial x_j}\right) + Fv = 0, \qquad \alpha_{ij} = \alpha_{ji} \tag{2}$$

and denote the elements of the inverse matrices by a^{ij} and α^{ij} respectively.

Let $u(x) = u(x_1, \cdots, x_n)$ be a solution of (1) which is nonnegative in D and vanishes on Γ. We exclude the trivial case $u \equiv 0$. Suppose that the coefficients of (1) and (2) satisfy the relations

$$\|\alpha^{ij} - a^{ij}\| \text{ is non-negative definite in } D, \tag{3}$$

$$f \leqq F \text{ in } D. \tag{4}$$

The theorem of Hartman and Wintner states that *if* (3) *and* (4) *hold then any solution of* (2) *must have a zero in* $D+\Gamma$.

We now consider in D two general linear elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + fu = 0, \qquad a_{ij} = a_{ji}, \tag{5}$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\alpha_{ij}\frac{\partial v}{\partial x_j}\right) + \sum_{i=1}^{n} \beta_i \frac{\partial v}{\partial x_i} + Fv = 0, \qquad \alpha_{ij} = \alpha_{ji} \tag{6}$$

and determine conditions on the coefficients so that a Sturmian comparison theorem shall be valid. For simplicity we consider the case of two independent variables, the result in more than two vari-

Received by the editors July 26, 1958.

[1] This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 49(638)-398.

ables being a straightforward extension. That is, we consider the equations

$$L_1 u \equiv (au_x)_x + (bu_x)_y + (bu_y)_x + (cu_y)_y + 2du_x + 2eu_y + fu = 0, \tag{7}$$

$$L_2 v \equiv (\alpha v_x)_x + (\beta v_x)_y + (\beta v_y)_x + (\gamma v_y)_y + 2\delta v_x + 2\epsilon v_y + Fv = 0. \tag{8}$$

An application of Green's theorem to (7) for a function u which vanishes on Γ yields

$$-\iint_D uL_1 u\,dxdy = \iint_D [au_x^2 + 2bu_xu_y + cu_y^2 - 2duu_x - 2euu_y - fu^2]dxdy.$$

We now employ a device of Picard [3] extended by Hartman and Wintner [1]. The same technique was also used by Ou and Ding [2] and the author in connection with certain problems in equations of mixed type. Let P and Q be arbitrary smooth functions in D. Then for any function u vanishing on Γ we have

$$\iint_D \left[\frac{\partial}{\partial x}(Pu^2) + \frac{\partial}{\partial y}(Qu^2)\right]dxdy = 0.$$

Thus if in addition u is a solution of (7) we find

$$\iint_D [au_x^2 + 2bu_xu_y + cu_y^2 + 2(P-d)uu_x + 2(Q-e)uu_y + (P_x + Q_y - f)u^2]dxdy = 0. \tag{9}$$

The condition that the integrand be a positive semi-definite form in u, u_x, u_y is

$$(ac - b^2)(P_x + Q_y - f) \geqq a(Q-e)^2 - 2b(P-d)(Q-e) + c(P-d)^2. \tag{10}$$

Define

$$P = \frac{-\alpha v_x - \beta v_y}{v} + d - \xi, \qquad Q = \frac{-\beta v_x - \gamma v_y}{v} + e - \eta$$

where v is a positive solution of (8) and ξ, η are functions yet to be determined. Substitution for P and Q in (10) yields the expression

$$
\begin{aligned}
&(ac - b^2)(\alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2) \\
&\qquad + (ac - b^2)v[2\delta v_x + 2\epsilon v_y + (F - f - \xi_x - \eta_y + d_x + e_y)v] \\
&\qquad \geqq (a\beta^2 - 2b\alpha\beta + c\alpha^2)v_x^2 + 2(a\beta\gamma - b\beta^2 - b\alpha\gamma + c\alpha\beta)v_x v_y \\
&\qquad + (a\gamma^2 - 2b\beta\gamma + c\beta^2)v_y^2 + 2(a\beta\eta + c\alpha\xi - b\alpha\eta - b\beta\xi)vv_x \\
&\qquad + 2(a\eta\gamma + c\beta\xi - b\beta\eta - b\gamma\xi)vv_y \\
&\qquad + (a\eta^2 - 2b\xi\eta + c\xi^2)v^2.
\end{aligned} \tag{11}
$$

We now investigate conditions on the coefficients of (7) and (8) which will make the above inequality hold throughout D. This will contradict (9) and hence v must have a zero in D. If condition (3) is satisfied, which in this case becomes

$$a > \alpha, \qquad (a - \alpha)(c - \gamma) - (b - \beta)^2 \geqq 0, \tag{12}$$

then (11) will be a consequence of the inequality

$$
\begin{aligned}
(ac - b^2)v[2\delta v_x + 2\epsilon v_y + (F - f - \xi_x - \eta_y + d_x + e_y)v] \\
\geqq 2(a\beta\eta + c\alpha\xi - b\alpha\eta - b\beta\xi)vv_x \\
+ 2(a\eta\gamma + c\beta\xi - b\beta\eta - b\gamma\xi)vv_y + (a\eta^2 - 2b\xi\eta + c\xi^2)v^2.
\end{aligned}
$$

We select ξ, η to be solutions of the linear equations

$$
\begin{aligned}
(a\beta - b\alpha)\eta + (c\alpha - b\beta)\xi &= -\delta(ac - b^2), \\
(a\gamma - b\beta)\eta + (c\beta - b\gamma)\xi &= -\epsilon(ac - b^2).
\end{aligned}
$$

That this is always possible follows from the ellipticity of the operators L_1 and L_2. Hence (11) is now a consequence of

$$(ac - b^2)(F - f - \xi_x - \eta_y + d_x + e_y) - (a\eta^2 - 2b\xi\eta + c\xi^2) \geqq 0. \tag{13}$$

We therefore have the result: *if the coefficients of* (7) *and* (8) *satisfy relations* (12) *and* (13) *and if there is a non-negative solution of* (7), *not identically zero, which vanishes on* Γ, *then every solution of* (8) *must have a zero in* $D+\Gamma$.

In the special case $\delta = \epsilon \equiv 0$ we have $\xi = \eta \equiv 0$ and condition (13) becomes

$$F - f + d_x + e_y \geqq 0.$$

This is particularly useful if a comparison is desired between a general linear second order equation and the equation

$$\Delta u + mu = 0$$

m a constant. Then we have the conditions

$$a > 1, \qquad (a-1)(c-1) - b^2 \geqq 0,$$
$$m \geqq -d_x - e_y + f.$$

Of course a more general relation between the coefficients, sufficient to yield the conclusion of the theorem, can be obtained by imposing the condition that (11) be a positive semi-definite form in v, v_x, v_y.

Bibliography

1. P. Hartman and A. Wintner, *On a comparison theorem for self-adjoint partial differential equations of elliptic type*, Proc. Amer. Math. Soc. vol. 6 (1955) p. 862.

2. Sing-Mo Ou and Shia-Shi Ding, *Sur l'unicité du problème de Tricomi de l'équation de Chaplygin*, Acta Math. Sinica vol. 5 (1955) p. 393.

3. E. Picard, *Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives*, J. Math. Pures Appl. vol. 6 (1890) p. 145 ff.

University of California, Berkeley