SOME CONDITIONS UNDER WHICH A HOMOGENEOUS CONTINUUM IS A SIMPLE CLOSED CURVE[1]

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In a recent paper [3], the author added a note in proof stating that two of the results could be strengthened by using the fact that a nondegenerate continuous curve is a simple closed curve if it is nearly homogeneous and is not a triod. It is the main purpose of this note to present a proof of this theorem and to state stronger forms of two results in [3]. Also, a theorem is presented that is related to a question raised by Knaster and Kuratowski [6]. This question as to whether every nondegenerate homogeneous bounded plane continuum is a simple closed curve has been settled negatively with examples by Bing [1] and Bing and Jones [2]. Additional conditions under which there is an affirmative answer have been given in some of the references cited in [3]. It apparently has not been noticed previously that there is an affirmative answer for nondegenerate bounded continua that are homogeneously embedded in the plane. This result (Theorem 3) follows directly from a characterization of homogeneous decomposable bounded plane continua given by Jones [5] and the nonaccessibility of certain points of indecomposable plane continua [7].

Definitions. A *continuous curve* is a compact metric space that is connected and locally connected. A *triod* is a continuum which is separated into three mutually separated sets by one of its subcontinua. A continuum M is: (i) *homogeneous* if for any two points x and y of M there is a homeomorphism of M onto itself that carries x into y; (ii) *nearly homogeneous* if for any point x of M and any subset D of M, open relative to M, there is a homeomorphism of M onto itself that carries x into a point of D; (iii) *2-homogeneous* if for any two points x_1 and x_2 of M and any two points y_1 and y_2 of M there is a homeomorphism of M onto itself that carries x_1+x_2 onto y_1+y_2; (iv) *nearly 2-homogeneous* if for any two points x_1 and x_2 of M and any two subsets D_1 and D_2 of M that are open relative to M there exist two points y_1 and y_2 in D_1 and D_2, respectively, and a homeomorphism of M onto itself that carries x_1+x_2 onto y_1+y_2; (v) *homogeneously embedded* in a space S if for any two points x and y of M there is a homeomorphism of S onto itself that carries x into y and M onto itself.

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THEOREM 1. *If the nondegenerate continuous curve M is nearly homogeneous and is not a triod, then M is a simple closed curve.*

PROOF. There exist a closed subset K of M and two points p_1 and p_2 of M such that K is irreducible with respect to the property of separating p_1 from p_2 in M and $M-K$ is the sum of two mutually separated sets M_1 and M_2 that contain p_1 and p_2, respectively. Let $G_1, G_2, G_3, \cdots$ be a sequence of finite collections of connected open subsets of M such that for each i, (1) G_i covers K, (2) each element of G_i intersects K and has a diameter less than $1/i$, and (3) each element of G_{i+1} is a subset of some element of G_i. There is a finite collection T_1 of arcs such that (1) T_1^* is connected,[2] (2) each arc of T_1 intersects an element of G_1, and (3) each element of G_1 intersects an arc of T_1. Now define a sequence $T_1, T_2, T_3, \cdots$ of finite collections of arcs such that for each i $(i>1)$, (1) each arc of T_i is a subset of an element of G_{i-1} and intersects an arc of T_{i-1}, (2) each arc of T_i intersects an element of G_i, and (3) each element of G_i intersects an arc of T_i. Let K' denote the continuum $K+T_1^*+T_2^*+\cdots$.

Suppose that M is not a simple closed curve. That K' contains neither M_1 nor M_2 follows from the fact that a compact metric continuum is a simple closed curve provided it is nearly homogeneous and some arc in it contains a set that is open relative to that continuum. Hence $M-K'$ is the sum of two mutually separated sets M_1' and M_2', where $M_i'=M_i-K'$ $(i=1, 2)$. Since $K'-K$ is nowhere dense in M and each point of K is a limit point of both M_1 and M_2, it follows that each point of K is a limit point of both M_1' and M_2'.[3]

Now from the near-homogeneity of M, it follows that there is a homeomorphism f of M onto itself that carries some point x of K into M_1'. Then the point $f(x)$ is a limit point of both $f(M_1')$ and $f(M_2')$, so that M_1' intersects both $f(M_1')$ and $f(M_2')$. There exists an arc H in M such that $H+K'+f(K')$ is a continuum that is nowhere dense in M. Let $N=H+K'+f('K)$. Then $M-N$ is the sum of the three mutually separated sets $M_2'-N$, $M_1'\cdot f(M_1')-N$, and $M_1'\cdot f(M_2')-N$, and this is contrary to the hypothesis that M is not a triod. Hence M is a simple closed curve.

QUESTION. If the continuous curve M has no local separating point and p_1 and p_2 are two points of M, then does there exist a subcontinuum K of M such that (1) $M-K$ is the sum of two mutually

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[2] If L is a collection of point sets, then L^* denotes the set which is the sum of the elements of L.

[3] This method of constructing K' is similar to a method used by Zippin [8], but his result is not directly applicable here.

separated sets M_1 and M_2 containing p_1 and p_2, respectively, and (2) every point of K is a limit point of both M_1 and M_2?

THEOREM 2. *If the decomposable compact metric continuum M is nearly 2-homogeneous and is not a triod, then M is a simple closed curve.*

PROOF. It follows from Theorem 15 of [3] that M is a continuous curve. Since M is nearly homogeneous, it follows from Theorem 1 that M is a simple closed curve.

COROLLARY. *If the nondegenerate compact metric continuum M is 2-homogeneous and is not a triod, then M is a simple closed curve.*

THEOREM 3. *If the nondegenerate bounded continuum M is homogeneously embedded in a plane E, then M is a simple closed curve.*

PROOF. Suppose that M is not a simple closed curve. It follows from two results by F. B. Jones [4; 5] that some nondegenerate subcontinuum K of M is indecomposable. Mazurkiewicz [7] has shown that some point y of K is not accessible from the complement of K, and hence y is not accessible from the complement of M. Since some point x of M is accessible from the complement of M, this leads to the contradiction that there is no homeomorphism of E onto itself that carries x into y and M onto itself.

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